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DOI-HOPF MODULES AND YETTER-DRINFELD MODULES FOR QUASI-HOPF ALGEBRAS

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Abstract. For a quasi-Hopf algebra H, a left H-comodule algebra $\mathfrak{B}$ and a right H-module coalgebra C we will characterize the category of Doi-Hopf modules ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ in terms of modules. We will also show that for an H-bicomodule algebra $\mathbb{A}$ and an H-bimodule coalgebra C the category of generalized Yetter-Drinfeld modules ${}_{\mathbb{A}}\mathcal{Y}D(H)^C$ is isomorphic to a certain category of Doi-Hopf modules. Using this isomorphism we will transport the properties from the category of Doi-Hopf modules to the category of generalized Yetter-Drinfeld modules.

Introduction

Recall that the defining axioms for a quasi-bialgebra H are the same as for a bialgebra, with the coassociativity of the comultiplication replaced by a weaker property, called quasi-coassociativity: the comultiplication is coassociative up to conjugation by an invertible element $\Phi \in H \otimes H \otimes H$, called the reassociator. There are important differences with ordinary quasi-bialgebras: the definition of a quasi-bialgebra is not selfdual, and we cannot consider comodules over quasi-bialgebras, since they are not coassociative coalgebras. However, the category of (left or right) modules over a quasi-bialgebra is a monoidal category.

Using this categorical point of view, the category of relative Hopf modules has been introduced and studied in [6]. A right H-module coalgebra C is a coalgebra in the monoidal category $\mathcal{M}_H$, and a left $[C, H]$-Hopf module is a left C-comodule in the monoidal category $\mathcal{M}_H$. A generalization of this concept was presented in [3]: replacing the right H-action by an action of a left H-comodule algebra, we can define the notion of Doi-Hopf module over a quasi-bialgebra. At this point, we have to mention that there is a phylosophical problem with the introduction of H-comodule algebras: we cannot introduce them as algebras in the category of H-comodules, since this category does not exist, as we mentioned above. However, a formal definition of H-comodule (and H-bicomodule) algebra was given by Hausser and Nill in [14]. A more conceptual definition has been proposed in [3]. If $\mathfrak{A}$ is an associative algebra then the category of $(\mathfrak{A} \otimes H, \mathfrak{A})$-bimodules is monoidal. Moreover, $\mathfrak{A} \otimes H$ has a coalgebra structure within this monoidal category "compatible" with the unit element $1_{\mathfrak{A}} \otimes 1_H$ if and only if $\mathfrak{A}$ is a right H-comodule algebra (for the complete statement see Proposition 1.2 below). Of course, a similar result holds

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for a left H-comodule algebra (see Proposition 1.3). Moreover, if H is a quasi-Hopf algebra then any H-bicomodule algebra can be viewed in two different (but twist equivalent) ways as a left (right) H-comodule algebra.

The aim of this paper is to study the category of Doi-Hopf modules over a quasi-Hopf algebra H, and its connections to the category of Yetter-Drinfeld modules. If H is a quasi-bialgebra, $\mathfrak{B}$ a left H-comodule algebra and C a right H-module coalgebra then $\mathfrak{B}\otimes C$ is a $\mathfrak{B}$-coring (this means a coalgebra in the monoidal category of $\mathfrak{B}$-bimodules) and the category of right-left $(H,\mathfrak{B},C)$-Hopf modules, denoted by ${}^C\mathcal{M}(H)_{\mathfrak{B}}$, is isomorphic to the category of right comodules over the coring $\mathfrak{B}\otimes C$, conform [3, Theorem 5.4]. In particular, since $\mathfrak{B}\otimes C$ is flat as a left $\mathfrak{B}$-module we obtain that ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is a Grothendieck category.

It was shown in [3, Proposition 5.2] that the category ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the category of right modules over the generalized smash product $C^*\blacktriangleright\!\!<\mathfrak{B}$, if C is finite dimensional. In Section 2 we look at the case where C is infinite dimensional. Following the methods developed in [9, 11, 19], we will present two characterizations of the category of Doi-Hopf modules ${}^C\mathcal{M}(H)_{\mathfrak{B}}$. We first introduce the notion of rational (right) $C^*\blacktriangleright\!\!<\mathfrak{B}$-module, and then we will show that the category ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to $\mathrm{Rat}(\mathcal{M}_{C^*\blacktriangleright\!<\mathfrak{B}})$, the category of rational (right) $C^*\blacktriangleright\!\!<\mathfrak{B}$-modules. We notice that, in the coassociative case, the notion of rational (right) $C^*\blacktriangleright\!\!<\mathfrak{B}$-module reduces to the notion of right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module which is rational as a right C^*-module. Secondly, we will show that $\mathrm{Rat}(\mathcal{M}_{C^*\blacktriangleright\!<\mathfrak{B}})$ is equal to $\sigma_{C^*\blacktriangleright\!<\mathfrak{B}}[C\otimes\mathfrak{B}]$, the smallest closed subcategory of $\mathcal{M}_{C^*\blacktriangleright\!<\mathfrak{B}}$ containing $C\otimes\mathfrak{B}$ (see Theorem 2.6). In this way we recover that ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is a Grothendieck category, a fortiori with enough injective objects. We will also introduce a generalized version of Koppinen's smash product [16], relate it to the generalized smash product, and characterize the category of Doi-Hopf modules as the full subcategory of modules over the Koppinen smash product, consisting of rational modules.

In Section 3, we will generalize a result from [8]. If H is a quasi-Hopf algebra, then an H-bicomodule algebra $\mathbb{A}$ can be viewed in two different, but twist equivalent, ways as a right $H^{\rm op}\otimes H$-comodule algebra. To this end, we first prove that any left H-comodule algebra $\mathfrak{B}$ can be turned into a right $H^{\rm op}$-comodule algebra. So, by this correspondence, the two (twist equivalent) left $H\otimes H^{\rm op}$-comodule algebra structures on $\mathbb{A}$ obtained in [7] provide two (twist equivalent) right $H^{\rm op}\otimes H$-comodule algebra structures on $\mathbb{A}$, which we will denote by $\mathbb{A}^1$ and $\mathbb{A}^2$. If C is an H-bimodule coalgebra (that is, a coalgebra in the monoidal category of H-bimodules), then C becomes in a natural way a left $H^{\rm op}\otimes H$-module coalgebra, thus it makes sense to consider the Hopf module category ${}_{\mathbb{A}^2}\mathcal{M}(H^{\rm op}\otimes H)^C$. The main result of Section 3 asserts that the category of generalized left-right Yetter-Drinfeld modules ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is isomorphic to ${}_{\mathbb{A}^2}\mathcal{M}(H^{\rm op}\otimes H)^C$, and also to ${}_{\mathbb{A}^1}\mathcal{M}(H^{\rm op}\otimes H)^C$. Using the first isomorphism, we will characterize ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ as a category of comodules over a coring. In Section 3.3, we will characterize the category of Yetter-Drinfeld modules as a category of modules.

1. Preliminary results

1.1. **Quasi-Hopf algebras.** We work over a commutative field k. All algebras, linear spaces etc. will be over k; unadorned $\otimes$ means $\otimes_k$. Following Drinfeld [12], a quasi-bialgebra is a fourtuple $(H,\Delta,\varepsilon,\Phi)$, where H is an associative algebra with unit, Φ is an invertible element in $H\otimes H\otimes H$, and $\Delta:\ H\to H\otimes H$ and $\varepsilon:\ H\to k$

are algebra homomorphisms satisfying the identities

$$(id\otimes\Delta)(\Delta(h)) = \Phi(\Delta\otimes id)(\Delta(h))\Phi^{-1}, \tag{1.1}$$

$$(id\otimes\varepsilon)(\Delta(h)) = h,\ (\varepsilon\otimes id)(\Delta(h)) = h, \tag{1.2}$$

for all $h\in H$; Φ has to be a normalized 3-cocycle, in the sense that

$$(1\otimes\Phi)(id\otimes\Delta\otimes id)(\Phi)(\Phi\otimes 1) = (id\otimes id\otimes\Delta)(\Phi)(\Delta\otimes id\otimes id)(\Phi), \tag{1.3}$$

$$(id\otimes\varepsilon\otimes id)(\Phi) = 1\otimes 1. \tag{1.4}$$

The map Δ is called the coproduct or the comultiplication, ε the counit and Φ the reassociator. As for Hopf algebras [20] we denote $\Delta(h) = h_1\otimes h_2$, but since Δ is only quasi-coassociative we adopt the further convention (summation implicitely understood):

$$(\Delta\otimes id)(\Delta(h)) = h_{(1,1)}\otimes h_{(1,2)}\otimes h_2,\ (id\otimes\Delta)(\Delta(h)) = h_1\otimes h_{(2,1)}\otimes h_{(2,2)},$$

for all $h\in H$. We will denote the tensor components of Φ by capital letters, and those of Φ^{-1} by small letters, namely

$$\Phi = X^1\otimes X^2\otimes X^3 = T^1\otimes T^2\otimes T^3 = V^1\otimes V^2\otimes V^3 = \cdots$$
$$\Phi^{-1} = x^1\otimes x^2\otimes x^3 = t^1\otimes t^2\otimes t^3 = v^1\otimes v^2\otimes v^3 = \cdots$$

A quasi-Hopf algebra is a quasi-bialgebra H equipped with an anti-automorphism S of the algebra H and elements $\alpha,\beta\in H$ such that

$$S(h_1)\alpha h_2 = \varepsilon(h)\alpha \text{ and } h_1\beta S(h_2) = \varepsilon(h)\beta, \tag{1.5}$$

$$X^1\beta S(X^2)\alpha X^3 = 1 \text{ and } S(x^1)\alpha x^2\beta S(x^3) = 1., \tag{1.6}$$

for all $h\in H$.

The antipode of a quasi-Hopf algebra is determined uniquely up to a transformation $\alpha\mapsto U\alpha$, $\beta\mapsto\beta U^{-1}$, $S(h)\mapsto US(h)U^{-1}$, with $U\in H$ invertible. The axioms imply that $\varepsilon(\alpha)\varepsilon(\beta)=1$, so, by rescaling α and β, we may assume without loss of generality that $\varepsilon(\alpha)=\varepsilon(\beta)=1$ and $\varepsilon\circ S=\varepsilon$. The identities (1.2-1.4) also imply that

$$(\varepsilon\otimes id\otimes id)(\Phi) = (id\otimes id\otimes\varepsilon)(\Phi) = 1\otimes 1.$$

If $H=(H,\Delta,\varepsilon,\Phi,S,\alpha,\beta)$ is a quasi-bialgebra or a quasi-Hopf algebra then $H^{\rm op}$, $H^{\rm cop}$ and $H^{\rm op,cop}$ are also quasi-bialgebras (respectively quasi-Hopf algebras), where the superscript "op" means opposite multiplication and "cop" means opposite comultiplication. The structure maps are obtained by putting $\Phi_{\rm op}=\Phi^{-1}$, $\Phi_{\rm cop}=(\Phi^{-1})^{321}$, $\Phi_{\rm op,cop}=\Phi^{321}$, $S_{\rm op}=S_{\rm cop}=(S_{\rm op,cop})^{-1}=S^{-1}$, $\alpha_{\rm op}=S^{-1}(\beta)$, $\beta_{\rm op}=S^{-1}(\alpha)$, $\alpha_{\rm cop}=S^{-1}(\alpha)$, $\beta_{\rm cop}=S^{-1}(\beta)$, $\alpha_{\rm op,cop}=\beta$ and $\beta_{\rm op,cop}=\alpha$.

The definition of a quasi-bialgebra H is designed in such a way that the categories of left and right representations over H are monoidal (see [15, 18] for the terminology). Let $(H,\Delta,\varepsilon,\Phi)$ be a quasi-bialgebra. For are left (resp. right) H-modules U,V,W, the associativity constraints $a_{U,V,W}$ (resp. $\mathbf{a}_{U,V,W}$) : $(U\otimes V)\otimes W\to U\otimes(V\otimes W)$ are given by the formulas

$$a_{U,V,W}((u\otimes v)\otimes w) = \Phi\cdot(u\otimes(v\otimes w));$$

$$\mathbf{a}_{U,V,W}((u\otimes v)\otimes w) = (u\otimes(v\otimes w))\cdot\Phi^{-1}.$$

Next we recall that the definition of a quasi-bialgebra or quasi-Hopf algebra is "twist covariant" in the following sense. An invertible element $F\in H\otimes H$ is called a gauge transformation or twist if $(\varepsilon\otimes id)(F)=(id\otimes\varepsilon)(F)=1$. If H is

a quasi-bialgebra or a quasi-Hopf algebra and $F = F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = G^1 \otimes G^2$, then we can define a new quasi-bialgebra (respectively quasi-Hopf algebra) H_F by keeping the multiplication, unit, counit (and antipode in the case of a quasi-Hopf algebra) of H and replacing the comultiplication, reassociator and the elements α and β by

$$\Delta_F(h) = F\Delta(h)F^{-1}, \tag{1.7}$$

$$\Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1), \tag{1.8}$$

$$\alpha_F = S(G^1)\alpha G^2, \ \beta_F = F^1\beta S(F^2). \tag{1.9}$$

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. For a quasi-Hopf algebra, we have the following statement: there exists a gauge transformation $f \in H \otimes H$ such that

$$f\Delta(S(h))f^{-1} = (S \otimes S)(\Delta^{\mathrm{cop}}(h)), \text{ for all } h \in H. \tag{1.10}$$

The element f can be computed explicitly. First set

$$A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}),$$
$$B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1),$$

and then define $\gamma, \delta \in H \otimes H$ by

$$\gamma = S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \text{ and } \delta = B^1\beta S(B^4) \otimes B^2\beta S(B^3).$$

Then f and f^{-1} are given by the formulae

$$f = (S \otimes S)(\Delta^{\mathrm{cop}}(x^1))\gamma\Delta(x^2\beta S(x^3)), \tag{1.11}$$

$$f^{-1} = \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{\mathrm{cop}}(x^3)). \tag{1.12}$$

Furthermore the corresponding twisted reassociator (see (1.8)) is given by

$$\Phi_f = (S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1). \tag{1.13}$$

1.2. **Comodule and bicomodule algebras.** A formal definition of comodule algebras over a quasi-bialgebra was given by Hausser and Nill [14].

Definition 1.1. Let H be a quasi-bialgebra. A right H-comodule algebra is a unital associative algebra $\mathfrak{A}$ together with an algebra morphism $\rho : \mathfrak{A} \to \mathfrak{A} \otimes H$ and an invertible element $\Phi_\rho \in \mathfrak{A} \otimes H \otimes H$ such that:

$$\Phi_\rho(\rho \otimes id)(\rho(\mathfrak{a})) = (id \otimes \Delta)(\rho(\mathfrak{a}))\Phi_\rho, \text{ for all } \mathfrak{a} \in \mathfrak{A}, \tag{1.14}$$

$$(1_{\mathfrak{A}} \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi_\rho)(\Phi_\rho \otimes 1_H) = (id \otimes id \otimes \Delta)(\Phi_\rho)(\rho \otimes id \otimes id)(\Phi_\rho), \tag{1.15}$$

$$(id \otimes \varepsilon) \circ \rho = id, \tag{1.16}$$

$$(id \otimes \varepsilon \otimes id)(\Phi_\rho) = (id \otimes id \otimes \varepsilon)(\Phi_\rho) = 1_{\mathfrak{A}} \otimes 1_H. \tag{1.17}$$

In a similar way, a left H-comodule algebra is a unital associative algebra $\mathfrak{B}$ together with an algebra morphism $\lambda : \mathfrak{B} \to H \otimes \mathfrak{B}$ and an invertible element $\Phi_\lambda \in H \otimes H \otimes \mathfrak{B}$

such that the following relations hold:

$$(id\otimes\lambda)(\lambda(\mathfrak{b}))\Phi_\lambda=\Phi_\lambda(\Delta\otimes id)(\lambda(\mathfrak{b})),\ \forall\ \mathfrak{b}\in\mathfrak{B}, \tag{1.18}$$

$$(1_H\otimes\Phi_\lambda)(id\otimes\Delta\otimes id)(\Phi_\lambda)(\Phi\otimes 1_{\mathfrak{B}}) = (id\otimes id\otimes\lambda)(\Phi_\lambda)(\Delta\otimes id\otimes id)(\Phi_\lambda), \tag{1.19}$$

$$(\varepsilon\otimes id)\circ\lambda=id, \tag{1.20}$$

$$(id\otimes\varepsilon\otimes id)(\Phi_\lambda)=(\varepsilon\otimes id\otimes id)(\Phi_\lambda)=1_H\otimes 1_{\mathfrak{B}}. \tag{1.21}$$

Observe that H is a left and right H-comodule algebra: take $\rho=\lambda=\Delta$, $\Phi_\rho=\Phi_\lambda=\Phi$.
If $(\mathfrak{B},\lambda,\Phi_\lambda)$ is a left H-comodule algebra then

- $(\mathfrak{B},\lambda\circ\tau_{H,\mathfrak{B}},(\Phi_\lambda^{-1})^{321})$ is a right H^{cop}-comodule algebra;
- $(\mathfrak{B}^{\mathrm{op}},\lambda\circ\tau_{H,\mathfrak{B}},\Phi_\lambda^{321})$ is a right $H^{\mathrm{op,cop}}$-comodule algebra;
- $(\mathfrak{B}^{\mathrm{op}},\lambda,\Phi_\lambda^{-1})$ is a left H^{op}-comodule algebra,

and vice versa. $\tau_{X,Y}:\ X\otimes Y\to Y\otimes X$ is the switch map mapping $x\otimes y$ to $y\otimes x$.
From [14], we recall the notion of twist equivalence for the coaction on a right H-comodule algebra $(\mathfrak{A},\rho,\Phi_\rho)$: if $\mathbb{V}\in\mathfrak{A}\otimes H$ is an invertible element such that

$$(id_{\mathfrak{A}}\otimes\varepsilon)(\mathbb{V})=1_{\mathfrak{A}} \tag{1.22}$$

then we can construct a new right H-comodule algebra $(\mathfrak{A},\rho',\Phi_{\rho'})$ with

$$\rho'(\mathfrak{a})=\mathbb{V}\rho(\mathfrak{a})\mathbb{V}^{-1} \tag{1.23}$$

and

$$\Phi_{\rho'}=(id_{\mathfrak{A}}\otimes\Delta)(\mathbb{V})\Phi_\rho(\rho\otimes id_H)(\mathbb{V}^{-1})(\mathbb{V}^{-1}\otimes 1_H). \tag{1.24}$$

We say that $(\mathfrak{A},\rho,\Phi_\rho)$ and $(\mathfrak{A},\rho',\Phi'_\rho)$ are twist equivalent right H-comodule algebras.
We obviously have a similar notion for left H-comodule algebras. More precisely, if $(\mathfrak{B},\lambda,\Phi_\lambda)$ is a left H-comodule algebra and $\mathbb{U}\in H\otimes\mathfrak{B}$ is an invertible element such that $(\varepsilon\otimes id_{\mathfrak{B}})(\mathbb{U})=1_{\mathfrak{B}}$ then we have a new left H-comodule algebra $(\mathfrak{B},\lambda',\Phi_{\lambda'})$ with $\lambda'(\mathfrak{b})=\mathbb{U}\lambda(\mathfrak{b})\mathbb{U}^{-1}$, for all $\mathfrak{b}\in\mathfrak{B}$, and

$$\Phi_{\lambda'}=(1_H\otimes\mathbb{U})(id_H\otimes\lambda)(\mathbb{U})\Phi_\lambda(\Delta\otimes id_{\mathfrak{B}})\mathbb{U}^{-1}.$$

$(\mathfrak{B},\lambda,\Phi_\lambda)$ and $(\mathfrak{B},\lambda',\Phi'_\lambda)$ are called twist equivalent.
For a right H-comodule algebra $(\mathfrak{A},\rho,\Phi_\rho)$ we will use the following Sweedler-type notation, for any $\mathfrak{a}\in\mathfrak{A}$.

$$\rho(\mathfrak{a})=\mathfrak{a}_{\langle 0\rangle}\otimes\mathfrak{a}_{\langle 1\rangle},\ (\rho\otimes id)(\rho(\mathfrak{a}))=\mathfrak{a}_{\langle 0,0\rangle}\otimes\mathfrak{a}_{\langle 0,1\rangle}\otimes\mathfrak{a}_{\langle 1\rangle}\text{ etc.}$$

Similarly, for a left H-comodule algebra $(\mathfrak{B},\lambda,\Phi_\lambda)$, if $\mathfrak{b}\in\mathfrak{B}$, we adopt the following notation:

$$\lambda(\mathfrak{b})=\mathfrak{b}_{[-1]}\otimes\mathfrak{b}_{[0]},\ (id\otimes\lambda)(\lambda(\mathfrak{b}))=\mathfrak{b}_{[-1]}\otimes\mathfrak{b}_{[0,-1]}\otimes\mathfrak{b}_{[0,0]}\text{ etc.}$$

In analogy with the notation for the reassociator Φ of H, we will write

$$\Phi_\rho=\tilde{X}^1_\rho\otimes\tilde{X}^2_\rho\otimes\tilde{X}^3_\rho=\tilde{Y}^1_\rho\otimes\tilde{Y}^2_\rho\otimes\tilde{Y}^3_\rho=\cdots$$

and

$$\Phi_\rho^{-1}=\tilde{x}^1_\rho\otimes\tilde{x}^2_\rho\otimes\tilde{x}^3_\rho=\tilde{y}^1_\rho\otimes\tilde{y}^2_\rho\otimes\tilde{y}^3_\rho=\cdots.$$

We use a similar notation for the element Φ_λ of a left H-comodule algebra $\mathfrak{B}$.

If H is an associative bialgebra and $\mathfrak{A}$ is an ordinary right H-comodule algebra, then $\mathfrak{A}\otimes H$ is a coalgebra in the monoidal category of $\mathfrak{A}$-bimodules. The quasi-bialgebra analog of this property was given in [3]. Let H be a quasi-bialgebra and $\mathfrak{A}$ a unital associative algebra. We define by ${}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$ the category of $\mathfrak{A}$-bimodules and $(H,\mathfrak{A})$-bimodules M such that $h\cdot(\mathfrak{a}\cdot m)=\mathfrak{a}\cdot(h\cdot m)$, for all $\mathfrak{a}\in\mathfrak{A}$, $h\in H$ and $m\in M$. Morphisms are left H-linear maps which are also $\mathfrak{A}$-bimodule maps. It is not hard to see that ${}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$ is a monoidal category. The tensor product is $\otimes_{\mathfrak{A}}$ and for any two objects $M,N\in {}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$, $M\otimes_{\mathfrak{A}}N$ is an object of ${}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$ via

$$(\mathfrak{a}\otimes h)\cdot(m\otimes_{\mathfrak{A}}n)\cdot\mathfrak{a}'=\mathfrak{a}\cdot(h_1\cdot m)\otimes_{\mathfrak{A}}h_2\cdot n\cdot\mathfrak{a}'$$

for all $m\in M$, $n\in N$, $\mathfrak{a},\mathfrak{a}'\in\mathfrak{A}$, and $h\in H$. The associativity constraints are given by

$$\begin{aligned}&\underline{a}_{M,N,P}:(M\otimes_{\mathfrak{A}}N)\otimes_{\mathfrak{A}}P\to M\otimes_{\mathfrak{A}}(N\otimes_{\mathfrak{A}}P),\\&\underline{a}_{M,N,P}((m\otimes_{\mathfrak{A}}n)\otimes_{\mathfrak{A}}p)=X^1\cdot m\otimes_{\mathfrak{A}}(X^2\cdot n\otimes_{\mathfrak{A}}X^3\cdot p);\end{aligned}$$

the unit object is $\mathfrak{A}$ viewed as a trivial left H-module, and the left and right unit constraints are the usual ones. Now, the definition of a comodule algebra in terms of monoidal categories can be restated as follows.

Proposition 1.2. ([3, Proposition 3.8]) *Let H be a quasi-bialgebra and $\mathfrak{A}$ an algebra, and view $\mathfrak{A}\otimes H$ in the canonical way as an object in ${}_{\mathfrak{A}\otimes H}\mathcal{M}$. There is a bijective correspondence between coalgebra structures $(\mathfrak{A}\otimes H,\underline{\Delta},\underline{\varepsilon})$ in the monoidal category ${}_{\mathfrak{A}\otimes H}\mathcal{M}_{\mathfrak{A}}$ such that $\underline{\Delta}(1_{\mathfrak{A}}\otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{A}}\otimes 1_H)=1_{\mathfrak{A}}$, and right H-comodule algebra structures on $\mathfrak{A}$.*

A similar result holds for a left H-comodule algebra $\mathfrak{B}$. Denote by ${}_{\mathfrak{B}}\mathcal{M}_{\mathfrak{B}\otimes H}$ the category whose objects are $\mathfrak{B}$-bimodules and $(\mathfrak{B},H)$-bimodules M such that $(m\cdot\mathfrak{b})\cdot h=(m\cdot h)\cdot\mathfrak{b}$ for all $m\in M$, $\mathfrak{b}\in\mathfrak{B}$ and $h\in H$. Morphisms are right H-linear maps which are also $\mathfrak{B}$-bimodule maps. We can easily check that ${}_{\mathfrak{B}}\mathcal{M}_{\mathfrak{B}\otimes H}$ is a monoidal category with tensor product $\otimes_{\mathfrak{B}}$ given via Δ, this means

$$\mathfrak{b}\cdot(m\otimes_{\mathfrak{B}}n)\cdot(\mathfrak{b}'\otimes h):=\mathfrak{b}\cdot m\cdot h_1\otimes_{\mathfrak{B}}(n\cdot h_2)\cdot\mathfrak{b}'$$

for all $M,N\in {}_{\mathfrak{B}}\mathcal{M}_{\mathfrak{B}\otimes H}$, $m\in M$, $n\in N$, $\mathfrak{b},\mathfrak{b}'\in\mathfrak{B}$ and $h\in H$. The associativity constraints are given by

$$\begin{aligned}&\underline{a}'_{M,N,P}:\ (M\otimes_{\mathfrak{B}}N)\otimes_{\mathfrak{B}}P\to M\otimes_{\mathfrak{B}}(N\otimes_{\mathfrak{B}}P),\\&\underline{a}'_{M,N,P}((m\otimes_{\mathfrak{B}}n)\otimes_{\mathfrak{B}}p)=m\cdot x^1\otimes_{\mathfrak{B}}(n\cdot x^2\otimes_{\mathfrak{B}}p\cdot x^3),\end{aligned}$$

the unit object is $\mathfrak{B}$ viewed as a right H-module via ε, and the left and right unit constraints are the usual ones.

Proposition 1.3. *Let H be a quasi-bialgebra and $\mathfrak{B}$ an algebra, and view $\mathfrak{B}\otimes H$ in the canonical way as an object in $\mathcal{M}_{\mathfrak{B}\otimes H}$. There is a bijective correspondence between coalgebra structures $(\mathfrak{B}\otimes H,\underline{\Delta},\underline{\varepsilon})$ in the monoidal category ${}_{\mathfrak{B}}\mathcal{M}_{\mathfrak{B}\otimes H}$ such that $\underline{\Delta}(1_{\mathfrak{B}}\otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{B}}\otimes 1_H)=1_{\mathfrak{B}}$, and left H-comodule algebra structures on $\mathfrak{B}$.*

Proof. Since the proof is similar to the one given in [3, Proposition 3.8], we restrict ourselves to a description of the correspondence.
Suppose that $(\mathfrak{B}\otimes H,\underline{\Delta},\underline{\varepsilon})$ is a coalgebra in ${}_{\mathfrak{B}}\mathcal{M}_{\mathfrak{B}\otimes H}$ such that $\underline{\Delta}(1_{\mathfrak{B}}\otimes 1_H)$ is invertible and $\underline{\varepsilon}(1_{\mathfrak{B}}\otimes 1_H)=1_{\mathfrak{B}}$. Write

$$\underline{\Delta}(1_{\mathfrak{B}}\otimes 1_H)=(1_{\mathfrak{B}}\otimes\tilde{X}^1_\lambda)\otimes_{\mathfrak{B}}(\tilde{X}^3_\lambda\otimes\tilde{X}^2_\lambda),$$

and consider $\Phi_\lambda = \tilde{X}^1_\lambda \otimes \tilde{X}^2_\lambda \otimes \tilde{X}^3_\lambda$. Define $\lambda : \to H \otimes \mathfrak{B}$ by

$$\lambda(b) = \tau_{\mathfrak{B},H}(\mathfrak{b} \cdot (1_{\mathfrak{B}} \otimes 1_H)),$$

for all $\mathfrak{b} \in \mathfrak{B}$. Then $(\mathfrak{B}, \lambda, \Phi_\lambda)$ is a left H-comodule algebra.
Conversely, if $(\mathfrak{B}, \lambda, \Phi_\lambda)$ is a left H-comodule algebra then $\mathfrak{B} \otimes H \in {}_{\mathfrak{B}}\mathcal{M}_{\mathfrak{B}\otimes H}$ via

$$\mathfrak{b} \cdot (\mathfrak{b}' \otimes h) \cdot (\mathfrak{b}'' \otimes h') = \mathfrak{b}_{[0]}\mathfrak{b}'\mathfrak{b}'' \otimes \mathfrak{b}_{[-1]}hh'.$$

Moreover, $\mathfrak{B}\otimes H$ is a coalgebra in ${}_{\mathfrak{B}}\mathcal{M}_{\mathfrak{B}\otimes H}$, with comultiplication and counit given by the formulas

$$\underline{\Delta}(\mathfrak{b} \otimes h) = (1_{\mathfrak{B}} \otimes \tilde{X}^1_\lambda h_1) \otimes_{\mathfrak{B}} (\tilde{X}^3_\lambda \mathfrak{b} \otimes \tilde{X}^2_\lambda h_2),$$

$$\underline{\varepsilon}(\mathfrak{b} \otimes h) = \varepsilon(h)\mathfrak{b},$$

for all $\mathfrak{b} \in \mathfrak{B}$ and $h \in H$. $\square$

Let $\mathfrak{B}$ be a left H-comodule algebra, and consider the elements $\tilde{p}_\lambda$ and $\tilde{q}_\lambda$ in $H \otimes \mathfrak{B}$ given by the following formulas:

$$\tilde{p}_\lambda = \tilde{p}^1_\lambda \otimes \tilde{p}^2_\lambda = \tilde{X}^2_\lambda S^{-1}(\tilde{X}^1_\lambda \beta) \otimes \tilde{X}^3_\lambda,\ \tilde{q}_\lambda = S(\tilde{x}^1_\lambda)\alpha\tilde{x}^2_\lambda \otimes \tilde{x}^3_\lambda. \tag{1.25}$$

Then we have the following formulas, for all $\mathfrak{b} \in \mathfrak{B}$ (see [14]):

$$\lambda(\mathfrak{b}_{[0]})\tilde{p}_\lambda[S^{-1}(\mathfrak{b}_{[-1]}) \otimes 1_{\mathfrak{B}}] = \tilde{p}_\lambda[1_H \otimes \mathfrak{b}], \tag{1.26}$$

$$[S(\mathfrak{b}_{[-1]}) \otimes 1_H]\tilde{q}_\lambda \lambda(\mathfrak{b}_{[0]}) = [1 \otimes \mathfrak{b}]\tilde{q}_\lambda, \tag{1.27}$$

$$\lambda(\tilde{q}^2_\lambda)\tilde{p}_\lambda[S^{-1}(\tilde{q}^1_\lambda) \otimes 1_{\mathfrak{B}}] = 1_H \otimes 1_{\mathfrak{B}}, \tag{1.28}$$

$$[S(\tilde{p}^1_\lambda) \otimes 1_{\mathfrak{B}}]\tilde{q}_\lambda \lambda(\tilde{p}^2_\lambda) = 1_H \otimes 1_{\mathfrak{B}}, \tag{1.29}$$

$$\begin{aligned}&\Phi^{-1}_\lambda(id_{\mathfrak{B}} \otimes \lambda)(\tilde{p}_\lambda)(1_{\mathfrak{B}} \otimes \tilde{p}_\lambda)\\ &\qquad = (\Delta \otimes id_{\mathfrak{B}})(\lambda(\tilde{X}^3_\lambda)\tilde{p}_\lambda)[S^{-1}(\tilde{X}^2_\lambda g^2) \otimes S^{-1}(\tilde{X}^1_\lambda g^1) \otimes 1_{\mathfrak{B}}],\end{aligned} \tag{1.30}$$

$$\begin{aligned}&(1_H \otimes \tilde{q}_\lambda)(id_H \otimes \lambda)(\tilde{q}_\lambda)\Phi_\lambda\\ &\qquad = [S(\tilde{x}^2_\lambda) \otimes S(\tilde{x}^1_\lambda) \otimes 1_{\mathfrak{B}}][f \otimes 1_{\mathfrak{B}}](\Delta \otimes id_{\mathfrak{B}})(\tilde{q}_\lambda \lambda(\tilde{x}^3_\lambda)).\end{aligned} \tag{1.31}$$

Bicomodule algebras where introduced by Hausser and Nill in [14], under the name “quasi-commuting pair of H-coactions”.

Definition 1.4. Let H be a quasi-bialgebra. An H-bicomodule algebra $\mathbb{A}$ is a quintuple $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$, where λ and ρ are left and right H-coactions on $\mathbb{A}$, and where $\Phi_\lambda \in H \otimes H \otimes \mathbb{A}$, $\Phi_\rho \in \mathbb{A} \otimes H \otimes H$ and $\Phi_{\lambda,\rho} \in H \otimes \mathbb{A} \otimes H$ are invertible elements, such that

- $(\mathbb{A}, \lambda, \Phi_\lambda)$ is a left H-comodule algebra,
- $(\mathbb{A}, \rho, \Phi_\rho)$ is a right H-comodule algebra,
- the following compatibility relations hold, for all $u \in \mathbb{A}$:

$$\Phi_{\lambda,\rho}(\lambda \otimes id)(\rho(u)) = (id \otimes \rho)(\lambda(u))\Phi_{\lambda,\rho} \tag{1.32}$$

$$\begin{aligned}&(1_H \otimes \Phi_{\lambda,\rho})(id \otimes \lambda \otimes id)(\Phi_{\lambda,\rho})(\Phi_\lambda \otimes 1_H)\\ &\qquad = (id \otimes id \otimes \rho)(\Phi_\lambda)(\Delta \otimes id \otimes id)(\Phi_{\lambda,\rho})\end{aligned} \tag{1.33}$$

$$\begin{aligned}&(1_H \otimes \Phi_\rho)(id \otimes \rho \otimes id)(\Phi_{\lambda,\rho})(\Phi_{\lambda,\rho} \otimes 1_H)\\ &\qquad = (id \otimes id \otimes \Delta)(\Phi_{\lambda,\rho})(\lambda \otimes id \otimes id)(\Phi_\rho).\end{aligned} \tag{1.34}$$

It was pointed out in [14] that the following additional relations hold in an H-bicomodule algebra $\mathbb{A}$:

$$(id_H \otimes id_{\mathbb{A}} \otimes \varepsilon)(\Phi_{\lambda,\rho}) = 1_H \otimes 1_{\mathbb{A}},\ \ (\varepsilon \otimes id_{\mathbb{A}} \otimes id_H)(\Phi_{\lambda,\rho}) = 1_{\mathbb{A}} \otimes 1_H.$$

As a first example, take $\mathbb{A} = H$, $\lambda = \rho = \Delta$ and $\Phi_\lambda = \Phi_\rho = \Phi_{\lambda,\rho} = \Phi$.
Let $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ be an H-bicomodule algebra; it is not hard to show that

- $(\mathbb{A}, \rho\circ\tau, \lambda\circ\tau, (\Phi_\rho^{-1})^{321}, (\Phi_\lambda^{-1})^{321}, (\Phi_{\lambda,\rho}^{-1})^{321})$ is an $H^{\rm cop}$-bicomodule algebra,
- $(\mathbb{A}^{\rm op}, \rho\circ\tau, \lambda\circ\tau, \Phi_\rho^{321}, \Phi_\lambda^{321}, \Phi_{\lambda,\rho}^{321})$ is an $H^{\rm op,cop}$-bicomodule algebra,
- $(\mathbb{A}^{\rm op}, \lambda, \rho, \Phi_\lambda^{-1}, \Phi_\rho^{-1}, \Phi_{\lambda,\rho}^{-1})$ is an $H^{\rm op}$-bicomodule algebra.

We will use the following notation:

$$\Phi_{\lambda,\rho} = \Theta^1 \otimes \Theta^2 \otimes \Theta^3 = \overline{\Theta}^1 \otimes \overline{\Theta}^2 \otimes \overline{\Theta}^3;$$

$$\Phi_{\lambda,\rho}^{-1} = \theta^1 \otimes \theta^2 \otimes \theta^3 = \overline{\theta}^1 \otimes \overline{\theta}^2 \otimes \overline{\theta}^3.$$

Let H be a quasi-Hopf algebra and $\mathbb{A}$ an H-bicomodule algebra. We define two left $H \otimes H^{\rm op}$-coactions $\lambda_1, \lambda_2 : \mathbb{A} \to (H \otimes H^{\rm op}) \otimes \mathbb{A}$ on $\mathbb{A}$, as follows:

$$\begin{aligned} \lambda_1(u) &= \big(u_{<0>_{[-1]}} \otimes S^{-1}(u_{<1>})\big) \otimes u_{<0>_{[0]}}, \\ \lambda_2(u) &= \big(u_{[-1]} \otimes S^{-1}(u_{[0]_{<1>}})\big) \otimes u_{[0]_{<0>}}, \end{aligned}$$

for all $u \in \mathbb{A}$. We also consider the following elements $\Phi_{\lambda_1}, \Phi_{\lambda_2} \in (H \otimes H^{\rm op}) \otimes (H \otimes H^{\rm op}) \otimes \mathbb{A}$:

$$\begin{aligned} \Phi_{\lambda_1} = &\Big(\Theta^1 \tilde{X}_\lambda^1 (\tilde{x}_\rho^1)_{[-1]_1} \otimes S^{-1}(\tilde{x}_\rho^3 g^2)\Big) \\ &\otimes \Big(\Theta^2_{[-1]} \tilde{X}_\lambda^2 (\tilde{x}_\rho^1)_{[-1]_2} \otimes S^{-1}(\Theta^3 \tilde{x}_\rho^2 g^1)\Big) \otimes \Theta^2_{[0]} \tilde{X}_\lambda^3 (\tilde{x}_\rho^1)_{[0]}, \\ \Phi_{\lambda_2} = &\Big(\tilde{Y}_\lambda^1 \otimes S^{-1}(\theta^3 \tilde{y}_\rho^3 (\tilde{Y}_\lambda^3)_{<1>_2} g^2)\Big) \\ &\otimes \Big(\theta^1 \tilde{Y}_\lambda^2 \otimes S^{-1}(\theta^2_{<1>} \tilde{y}_\rho^2 (\tilde{Y}_\lambda^3)_{<1>_1} g^1)\Big) \otimes \theta^2_{<0>} \tilde{y}_\rho^1 (\tilde{Y}_\lambda^3)_{<0>}. \end{aligned}$$

It was proved in [7] that $(\mathbb{A}, \lambda_1, \Phi_{\lambda_1})$ and $(\mathbb{A}, \lambda_2, \Phi_2)$ are twist equivalent left $H \otimes H^{\rm op}$-comodule algebras. In particular, if H is a quasi-Hopf algebra then the notion of H-bicomodule algebra can be restated in terms of monoidal categories. In Section 3 we will see that $\mathbb{A}$ can be also viewed in two twist equivalent ways as a right $H^{\rm op} \otimes H$-comodule algebra.

2. Doi-Hopf modules and rationality properties

2.1. **Doi-Hopf modules.** The category of left (right) modules over a quasi-bialgebra is monoidal. A coalgebra in ${}_H\mathcal{M}$ (resp. $\mathcal{M}_H$) is called a left (right) H-module coalgebra. Thus a left H-module coalgebra is a left H-module C together with a comultiplication $\underline{\Delta} :\ C \to C \otimes C$ and a counit $\underline{\varepsilon} :\ C \to k$ such that

$$\Phi(\underline{\Delta} \otimes id_C)(\underline{\Delta}(c)) = (id_C \otimes \underline{\Delta})(\underline{\Delta}(c)), \tag{2.1}$$

$$\underline{\Delta}(h \cdot c) = h_1 \cdot c_{\underline{1}} \otimes h_2 \cdot c_{\underline{2}}, \tag{2.2}$$

$$\underline{\varepsilon}(h \cdot c) = \varepsilon(h)\underline{\varepsilon}(c), \tag{2.3}$$

for all $c \in C$ and $h \in H$. Similarly, a right H-module coalgebra C is a right H-module together with a comultiplication $\underline{\Delta} :\ C \to C \otimes C$ and a counit $\underline{\varepsilon} :\ C \to k$, satisfying the following relations

$$(\underline{\Delta} \otimes id_C)(\underline{\Delta}(c))\Phi^{-1} = (id_C \otimes \underline{\Delta})(\underline{\Delta}(c)), \tag{2.4}$$

$$\underline{\Delta}(c \cdot h) = c_{\underline{1}} \cdot h_1 \otimes c_{\underline{2}} \cdot h_2, \tag{2.5}$$

$$\underline{\varepsilon}(c \cdot h) = \underline{\varepsilon}(c)\varepsilon(h), \tag{2.6}$$

for all $c \in C$ and $h \in H$. Here we used the Sweedler-type notation

$$\underline{\Delta}(c) = c_{\underline{1}} \otimes c_{\underline{2}}, \quad (\underline{\Delta} \otimes id_C)(\underline{\Delta}(c)) = c_{(\underline{1},\underline{1})} \otimes c_{(\underline{1},\underline{2})} \otimes c_{\underline{2}} \quad \text{etc.}$$

It is easy to see that a left H-module coalgebra C is in a natural way a right H^{op}-module coalgebra (and vice versa).
Let H be a quasi-bialgebra and C a right H-module coalgebra. A left $[C,H]$-Hopf module is a left C-comodule in the monoidal category $\mathcal{M}_H$. This definition was generalized in [3].

Definition 2.1. Let H be a quasi-bialgebra over a field k, C a right H-module coalgebra and $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra. A right-left $(H, \mathfrak{B}, C)$-Hopf module (or Doi-Hopf module) is a k-module M, with the following additional structure: M is right $\mathfrak{B}$-module (the right action of $\mathfrak{b}$ on m is denoted by $m \cdot \mathfrak{b}$), and we have a k-linear map $\lambda_M : \ M \to C \otimes M$, such that the following relations hold, for all $m \in M$ and $\mathfrak{b} \in \mathfrak{B}$:

$$(\underline{\Delta} \otimes id_M)(\lambda_M(m)) = (id_C \otimes \lambda_M)(\lambda_M(m))\Phi_\lambda, \tag{2.7}$$

$$(\underline{\varepsilon} \otimes id_M)(\lambda_M(m)) = m, \tag{2.8}$$

$$\lambda_M(m \cdot \mathfrak{b}) = m_{\{-1\}} \cdot \mathfrak{b}_{[-1]} \otimes m_{\{0\}} \cdot \mathfrak{b}_{[0]}. \tag{2.9}$$

As usual, we use the Sweedler-type notation $\lambda_M(m) = m_{\{-1\}} \otimes m_{\{0\}}$. ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is the category of right-left $(H, \mathfrak{B}, C)$-Hopf modules and right $\mathfrak{B}$-linear, left C-colinear k-linear maps.

Let M be a right $\mathfrak{B}$-module; then $C \otimes M$ is a right-left $(H, \mathfrak{B}, C)$-Hopf module, with structure maps given by the following formulas

$$(c \otimes m) \cdot \mathfrak{b} = c \cdot \mathfrak{b}_{[-1]} \otimes m \cdot \mathfrak{b}_{[0]}, \tag{2.10}$$

$$\lambda_{C \otimes M}(c \otimes m) = c_{\underline{1}} \cdot \tilde{x}^1_\lambda \otimes c_{\underline{2}} \cdot \tilde{x}^2_\lambda \otimes m \cdot \tilde{x}^3_\lambda, \tag{2.11}$$

for all $c \in C$, $\mathfrak{b} \in \mathfrak{B}$ and $m \in M$. We obtain a functor $\mathcal{F} = C \otimes \bullet : \ \mathcal{M}_{\mathfrak{B}} \to {}^C\mathcal{M}(H)_{\mathfrak{B}}$. The functor $\mathcal{F}$ sends a morphism ϑ to $id_C \otimes \vartheta$. In particular, $C \otimes \mathfrak{B} \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$, via the structure maps

$$(c \otimes \mathfrak{b}) \cdot \mathfrak{b}' = c \cdot \mathfrak{b}'_{[-1]} \otimes \mathfrak{b}\mathfrak{b}'_{[0]},$$
$$\lambda_{C \otimes \mathfrak{B}}(c \otimes \mathfrak{b}) = c_{\underline{1}} \cdot \tilde{x}^1_\lambda \otimes c_{\underline{2}} \cdot \tilde{x}^2_\lambda \otimes \mathfrak{b}\tilde{x}^3_\lambda,$$

for all $c \in C$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$.
The functor $\mathcal{F}$ has a left and a right adjoint, so it is an exact functor.

Proposition 2.2. *Let H be a quasi-bialgebra, $\mathfrak{B}$ a left H-comodule algebra and C a right H-module coalgebra. Then the functor $\mathcal{F} = C \otimes \bullet$ is a right adjoint of the forgetful functor*

$${}^C\mathcal{U} : \ {}^C\mathcal{M}(H)_{\mathfrak{B}} \to \mathcal{M}_{\mathfrak{B}},$$

and a left adjoint of the functor

$$\mathrm{Hom}^C_{\mathfrak{B}}(C \otimes \mathfrak{B}, \bullet) : \ {}^C\mathcal{M}(H)_{\mathfrak{B}} \to \mathcal{M}_{\mathfrak{B}}$$

defined as follows. For $M \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$, $\mathrm{Hom}^C_{\mathfrak{B}}(C \otimes \mathfrak{B}, M)$ is a right $\mathfrak{B}$-module via the formula

$$(\eta \cdot \mathfrak{b})(c \otimes \mathfrak{b}') = \eta(c \otimes \mathfrak{b}\mathfrak{b}'),$$

for all $\eta \in \mathrm{Hom}^C_{\mathfrak{B}}(C \otimes \mathfrak{B}, M)$, $c \in C$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$. For a morphism $\kappa : \ M \to N$ in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$, we let

$$\mathrm{Hom}^C_{\mathfrak{B}}(C \otimes \mathfrak{B}, \kappa)(\upsilon) = \kappa \circ \upsilon,$$

for all $\upsilon \in \mathrm{Hom}^C_{\mathfrak{B}}(C \otimes \mathfrak{B}, M)$.

Proof. Let M be a right-left $(H, \mathfrak{B}, C)$-Hopf module and N a right $\mathfrak{B}$-module. Define

$$\xi_{M,N} : \mathrm{Hom}_{\mathfrak{B}}(M, N) \to \mathrm{Hom}^C_{\mathfrak{B}}(M, C \otimes N),\ \xi_{M,N}(\varsigma)(m) = m_{\{-1\}} \otimes \varsigma(m_{\{0\}}),$$

for all $\varsigma \in \mathrm{Hom}_{\mathfrak{B}}(M, N)$ and $m \in M$, and

$$\zeta_{M,N} : \mathrm{Hom}^C_{\mathfrak{B}}(M, C \otimes N) \to \mathrm{Hom}_{\mathfrak{B}}(M, N),\ \zeta_{M,N}(\chi)(m) = (\underline{\varepsilon} \otimes id_N)(\chi(m)),$$

for all $\chi \in \mathrm{Hom}^C_{\mathfrak{B}}(M, C \otimes N)$ and $m \in M$. It is not hard to see that $\xi_{M,N}$ and $\zeta_{M,N}$ are well-defined natural transformations that are inverse to each other.
For $M \in \mathcal{M}_{\mathfrak{B}}$ and $N \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$, we define

$$\xi'_{M,N} : \mathrm{Hom}^C_{\mathfrak{B}}(C \otimes M, N) \to \mathrm{Hom}_{\mathfrak{B}}(M, \mathrm{Hom}^C_{\mathfrak{B}}(C \otimes \mathfrak{B}, N))$$

by

$$\xi'_{M,N}(\varsigma')(m)(c \otimes \mathfrak{b}) = \varsigma'(c \otimes m \cdot \mathfrak{b})$$

and

$$\zeta'_{M,N} : \mathrm{Hom}_{\mathfrak{B}}(M, \mathrm{Hom}^C_{\mathfrak{B}}(C \otimes \mathfrak{B}, N)) \to \mathrm{Hom}^C_{\mathfrak{B}}(C \otimes M, N)$$

by

$$\zeta'_{M,N}(\chi')(c \otimes m) = \chi'(m)(c \otimes 1_{\mathfrak{B}}),$$

for all $\varsigma' \in Hom^C_{\mathfrak{B}}(C \otimes M, N)$, $\chi' \in Hom_{\mathfrak{B}}(M, Hom^C_{\mathfrak{B}}(C \otimes \mathfrak{B}, N))$, $m \in M$, $c \in C$ and $\mathfrak{b} \in \mathfrak{B}$. Then ξ' and ζ' are well-defined natural transformations that are inverse to each other. $\square$

2.2. **Doi-Hopf modules and comodules over a coring.** It was proved in [3] that the category of right-left Doi-Hopf modules is isomorphic to the category of right comodules over a suitable $\mathfrak{B}$-coring. For a general treatment of the theory of corings, we refer to [1, 2].
Let R be a ring with unit. An R-coring $\mathcal{C}$ is an R-bimodule together with two R-bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C}$ and $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$ such that the usual coassociativity and counit properties hold. A right $\mathcal{C}$-comodule is a right R-module together with a right R-linear map $\rho^r_M : M \to M \otimes_R \mathcal{C}$ such that

$$(\rho^r_M \otimes_R id_{\mathcal{C}}) \circ \rho^r_M = (id_M \otimes_R \Delta_{\mathcal{C}}) \circ \rho^r_M,$$
$$(id_M \otimes_R \varepsilon_{\mathcal{C}}) \circ \rho^r_M = id_M.$$

A map $\hbar : M \to N$ between two right $\mathcal{C}$-comodules is called right $\mathcal{C}$-colinear if $\hbar$ is right R-linear and $\rho^r_N \circ \hbar = (\hbar \otimes_R id_{\mathcal{C}}) \circ \rho^r_M$. $\mathcal{M}^{\mathcal{C}}$ will be the category of right $\mathcal{C}$-comodules and right $\mathcal{C}$-comodule maps. It is well-known that $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category (in particular with enough injective objects) if $\mathcal{C}$ is flat as a left R-module. The category ${}^{\mathcal{C}}\mathcal{M}$ of left $\mathcal{C}$-comodules and left $\mathcal{C}$-comodule maps can be introduced in a similar way.
Let H be a quasi-bialgebra, $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra and C a right H-module coalgebra. It was proved in [3] that $\mathcal{C} = \mathfrak{B} \otimes C$ is a $\mathfrak{B}$-coring. More precisely, $\mathcal{C}$ is a $\mathfrak{B}$-bimodule via

$$\mathfrak{b} \cdot (\mathfrak{b}' \otimes c) = \mathfrak{b}\mathfrak{b}' \otimes c \text{ and } (\mathfrak{b} \otimes c) \cdot \mathfrak{b}' = \mathfrak{b}\mathfrak{b}'_{[0]} \otimes c \cdot \mathfrak{b}'_{[-1]},$$

and the comultiplication and counit are given by

$$\Delta_{\mathcal{C}}(\mathfrak{b} \otimes c) = (\mathfrak{b}\tilde{x}^3_\lambda \otimes c_{\underline{2}} \cdot \tilde{x}^2_\lambda) \otimes_{\mathfrak{B}} (1_{\mathfrak{B}} \otimes c_{\underline{1}} \cdot \tilde{x}^1_\lambda)$$

and

$$\varepsilon_{\mathcal{C}}(c\otimes\mathfrak{b})=\underline{\varepsilon}(c)\mathfrak{b},$$

for all $\mathfrak{b},\mathfrak{b}'\in\mathfrak{B}$ and $c\in C$. Then we have an isomorphism of categories (see [3, Theorem 5.4])

$${}^{C}\mathcal{M}(H)_{\mathfrak{B}}\cong\mathcal{M}^{\mathcal{C}}.$$

$\mathcal{C}$ is free, and therefore flat, as a left $\mathfrak{B}$-module, and we conclude that ${}^{C}\mathcal{M}(H)_{\mathfrak{B}}\cong\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category, and has therefore enough injectives (see [13, Prop. 1.2] or [2, 18.14].

2.3. **Doi-Hopf modules and the generalized smash product.** We now want to discuss when the category of Doi-Hopf modules is isomorphic to a module category. In the case where C is finite dimensional, this was already done in [3, Proposition 5.2]. Let us explain this more precisely.
Let H be a quasi-bialgebra. A left H-module algebra A is an algebra in the monoidal category ${}_H\mathcal{M}$. This means that A is a left H-module with a multiplication $A\otimes A\to A$ and a unit element 1_A satisfying the following conditions:

$$(aa')a''=(X^1\cdot a)[(X^2\cdot a')(X^3\cdot a'')], \tag{2.12}$$

$$h\cdot(aa')=(h_1\cdot a)(h_2\cdot a'), \tag{2.13}$$

$$h\cdot 1_A=\varepsilon(h)1_A, \tag{2.14}$$

for all $a,a',a''\in A$ and $h\in H$. Following [3], we can define the generalized smash product of a left H-module algebra A and a left H-comodule algebra $\mathfrak{B}$: $A\blacktriangleright\!\!<\mathfrak{B}=A\otimes B$ as a vector space, with multiplication

$$(a\blacktriangleright\!\!<\mathfrak{b})(a'\blacktriangleright\!\!<\mathfrak{b}')=(\tilde{x}^1_\lambda\cdot a)(\tilde{x}^2_\lambda\mathfrak{b}_{[-1]}\cdot a')\blacktriangleright\!\!<\tilde{x}^3_\lambda\mathfrak{b}_{[0]}\mathfrak{b}',$$

for all $a,a'\in A$, $\mathfrak{b},\mathfrak{b}'\in\mathfrak{B}$. $A\blacktriangleright\!\!<\mathfrak{B}$ is an associative algebra with unit $1_A\blacktriangleright\!\!<1_B$.
The linear dual C^* of a right H-module coalgebra C is a left H-module algebra. The multiplication is the convolution, the unit is $\underline{\varepsilon}$, and the left H-action is given by the formula $(h\cdot c^*)(c)=c^*(c\cdot h)$, for all $h\in H$, $c^*\in C^*$ and $c\in C$. So we can consider the generalized smash product algebra $C^*\blacktriangleright\!\!<\mathfrak{B}$. Moreover, we have a functor

$$\mathfrak{G}:\ {}^{C}\mathcal{M}(H)_{\mathfrak{B}}\to\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}},\ \mathfrak{G}(M)=M,$$

with right $C^*\blacktriangleright\!\!<\mathfrak{B}$-action given by

$$m\cdot(c^*\blacktriangleright\!\!<\mathfrak{b})=c^*(m_{\{-1\}})m_{\{0\}}\cdot\mathfrak{b},$$

for all $m\in M$, $c^*\in C^*$ and $\mathfrak{b}\in\mathfrak{B}$. If C is finite dimensional, then $\mathfrak{G}$ is an isomorphism of categories (see [3]).
Now let C be infinite dimensional. We will show that the category ${}^{C}\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the category $\sigma_{C^*\blacktriangleright<\mathfrak{B}}[C\otimes\mathfrak{B}]$. Recall that if $\mathcal{A}$ is a Grothendieck category and M is an object of $\mathcal{A}$ then $\sigma_{\mathcal{A}}[M]$ is the class of all objects $N\in\mathcal{A}$ which are subgenerated by M, that is, N is a subobject of a quotient of directs sums of copies of M. It is well-known that $\sigma_{\mathcal{A}}[M]$ is the smallest closed subcategory of $\mathcal{A}$ containing M, and that for any closed subcategory $\mathcal{D}$ of $\mathcal{A}$ there exists an object M of $\mathcal{A}$ such that $\mathcal{D}=\sigma_{\mathcal{A}}[M]$. In particular, $\sigma_{\mathcal{A}}[M]$ is a Grothendieck category, and has a fortiori enough injective objects. For more detail, the reader is invited to consult [10, 21].

Let H be a quasi-bialgebra, C a right H-module coalgebra and $\mathfrak{B}$ a left H-comodule coalgebra. For a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module M we define the linear maps

$$\mu_M:\ M\to \mathrm{Hom}(C^*\otimes\mathfrak{B},M),\ \ \mu_M(m)(c^*\otimes\mathfrak{b})=m\cdot(c^*\blacktriangleright\!\!<\mathfrak{b}),$$
$$\nu_M:\ C\otimes M\to \mathrm{Hom}(C^*\otimes\mathfrak{B},M),\ \ \nu_M(c\otimes m)(c^*\otimes\mathfrak{b})=c^*(c)m\cdot(\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}),$$

for all $m\in M$, $c\in C$, $c^*\in C^*$ and $\mathfrak{b}\in\mathfrak{B}$. It is easily verified that μ_M and ν_M are injective linear maps.
Inspired by [9, 11] we propose the following

Definition 2.3. Let M be a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module. We say that M is rational if $\mathrm{Im}\,(\mu_M)\subseteq \mathrm{Im}\,(\nu_M)$. $\mathrm{Rat}(\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}})$ will be the full subcategory of $\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}}$ consisting of rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-modules.

Remark 2.4. If H is a coassociative bialgebra, $\mathfrak{B}$ a left H-comodule algebra and C a right H-module coalgebra in the usual sense, then a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module M is nothing else that a $C^*\blacktriangleright\!\!<\mathfrak{B}$-module wich is rational as a C^*-module. Here M is viewed as a right C^*-module via the canonical algebra map $C^*\hookrightarrow C^*\blacktriangleright\!\!<\mathfrak{B}$.

It follows easily from Definition 2.3 that a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module is rational if and only if for every $m\in M$ there exist two finite sets $\{c_i\}_i\subseteq C$ and $\{m_i\}_i\subseteq M$ such that

$$m\cdot(c^*\blacktriangleright\!\!<\mathfrak{b})=\sum_i c^*(c_i)m_i\cdot(\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}),\tag{2.15}$$

for all $c^*\in C^*$ and $\mathfrak{b}\in\mathfrak{B}$. If $\{c'_j\}_j\subseteq C$ and $\{m'_j\}_j\subseteq M$ are two other finite sets satisfying (2.15), then $\sum\limits_i c_i\otimes m_i=\sum\limits_j c'_j\otimes m'_j$, because of the injectivity of the map ν_M. So we have a well-defined map

$$\lambda_M:\ M\to C\otimes M,\ \ \lambda_M(m)=\sum_i c_i\otimes m_i,$$

for all $m\in M$. If C is finite dimensional, then any right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module is rational. Indeed, we take a finite dual basis $\{(c_i,c^i)\}_i$ of C and then consider for each $m\in M$ the finite sets $\{c_i\}_i\subseteq C$ and $\{m_i=m\cdot(c^i\blacktriangleright\!\!<1_{\mathfrak{B}})\}_i\subseteq M$.
We now summarize the properties of rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-modules.

Proposition 2.5. *Let H be a quasi-bialgebra, C a right H-module coalgebra and $\mathfrak{B}$ a left H-comodule algebra. Then:*

- i) *A cyclic submodule of a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module is finite dimensional.*
- ii) *If M is a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module and N is a $C^*\blacktriangleright\!\!<\mathfrak{B}$-submodule of M, then N and M/N are rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-modules.*
- iii) *If $(M_i)_{i\in I}$ is a family of rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-modules, then the direct sum $M=\bigoplus\limits_{i\in I}M_i$ in $\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}}$ is a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module.*
- iv) *To any right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module M we can associate a unique maximal rational submodule M^{rat}, which is equal to $\mu_M^{-1}(Im(\nu_M))$. It is also equal to the sum of all rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-submodule of M. We have a left exact functor*

$$\mathrm{Rat}:\ \mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}}\to\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}},\ \ \mathrm{Rat}(M)=M^{\mathrm{rat}}.$$

Proof. In fact this is a straightforward generalization of [20, Theorem 2.1.3] and [10, Theorem 2.2.6]. Consider an element m of a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module m. In what follows, $\{c_i\}_i\subseteq C$ and $\{m_i\}_i\subseteq M$ will then be two finite sets satisfying (2.15).

i) Let $m\cdot(C^*\blacktriangleright\!\!<\mathfrak{B})$ be a cyclic submodule of a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module M. $m\cdot(C^*\blacktriangleright\!\!<\mathfrak{B})$ is generated by the finite set $\{m_i\}_i$, so it is finite dimensional.
ii) Take $m\in N\subset M$. Choose the c_i in such a way that they are linearly independent. Fix j, and take $c^*\in C^*$ such that $c^*(c_i)=\delta_{i,j}$. Then N contains $m\cdot(c^*\blacktriangleright\!\!<1_{\mathfrak{B}})=\sum\limits_i c^*(c_i)m_i\cdot(\underline{\varepsilon}\blacktriangleright\!\!<1_{\mathfrak{B}})=m_j$, as needed.
Let $\overline{m}$ be the class in M/N represented by $m\in M$. For all $c^*\in C^*$ and $\mathfrak{b}\in\mathfrak{B}$, we have that $\overline{m}\cdot(c^*\blacktriangleright\!\!<\mathfrak{b})=\sum\limits_i c^*(c_i)\overline{m_i}\cdot(\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b})$, and it follows that M/N is a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module.
iii) Every $m\in M$ can be written in a unique way as

$$m=\sum_{j\in J}m_j,$$

with $J\subset I$ finite, and $m_j\in M_j$. Since M_j is a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module, there exist two finite sets $\{c^j_k\}_k\subseteq C$ and $\{m^j_k\}_k\subseteq M_j$ such that

$$m_j\cdot(c^*\blacktriangleright\!\!<\mathfrak{b})=\sum_k c^*(c^j_k)m^j_k\cdot(\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}),$$

for all $c^*\in C^*$ and $\mathfrak{b}\in\mathfrak{B}$. We therefore have that

$$m\cdot(c^*\blacktriangleright\!\!<\mathfrak{b})=\sum_{j\in J}m_j\cdot(c^*\blacktriangleright\!\!<\mathfrak{b})=\sum_{j\in J,k}c^*(c^j_k)m^j_k\cdot(\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}),$$

for all $c^*\in C^*$ and $\mathfrak{b}\in\mathfrak{B}$, and it follows that M is a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module.
iv) Let M be a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module. We define $M^{\mathrm{rat}}=\mu_M^{-1}(\mathrm{Im}\,(\nu_M))$.
We first prove that M^{rat} is a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module. Take $m\in M^{\mathrm{rat}}$. Then there exist two finite sets $\{c_i\}_i\subseteq C$ and $\{m_i\}_i\subseteq M$ such that $m\cdot(c^*\blacktriangleright\!\!<\mathfrak{b})=c^*(c_i)m_i\cdot(\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b})$, for all $c^*\in C^*$ and $\mathfrak{b}\in\mathfrak{B}$. Therefore:

$$\begin{aligned}
&(m\cdot(c^*\blacktriangleright\!\!<\mathfrak{b}))\cdot(d^*\blacktriangleright\!\!<\mathfrak{b}')=m\cdot((c^*\blacktriangleright\!\!<\mathfrak{b})(d^*\blacktriangleright\!\!<\mathfrak{b}'))\\
&\quad=\quad m\cdot((\tilde{x}^1_\lambda\cdot c^*)(\tilde{x}^2_\lambda\mathfrak{b}_{[-1]}\cdot d^*)\blacktriangleright\!\!<\tilde{x}^3_\lambda\mathfrak{b}_{[0]}\mathfrak{b}')\\
&\quad=\quad c^*((c_i)_{\underline{1}}\cdot\tilde{x}^1_\lambda)d^*((c_i)_{\underline{2}}\cdot\tilde{x}^2_\lambda\mathfrak{b}_{[-1]})(m_i\cdot(\underline{\varepsilon}\blacktriangleright\!\!<\tilde{x}^3_\lambda\mathfrak{b}_{[0]}))\cdot(\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}'),
\end{aligned}$$

for all $m\in M$, $c^*,d^*\in C^*$ and $\mathfrak{b},\mathfrak{b}'\in\mathfrak{B}$. Thus $m\cdot(c^*\blacktriangleright\!\!<\mathfrak{b})\in M^{\mathrm{rat}}$, hence M^{rat} is a $C^*\blacktriangleright\!\!<\mathfrak{B}$-submodule of M. Using an argument similar to the one in the first part of the proof of assertion ii), we can easily check that M^{rat} is a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module.
Let N be a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-submodule of M, this means $\mathrm{Im}\,(\mu_N)\subseteq\mathrm{Im}\,(\nu_N)$. Then

$$\mu_M(N)=\mu_N(N)\subseteq\mathrm{Im}\,(\nu_N)\subseteq\mathrm{Im}\,(\nu_M),$$

hence $N\subseteq\mu_M^{-1}(\mathrm{Im}\,(\nu_M))=M^{\mathrm{rat}}$, and we conclude that M^{rat} is the unique maximal rational submodule of M. Assertions ii) and iii) show that M^{rat} is also equal to the sum of all rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-submodules of M. The proof of the final assertion is identical to the proof of [10, Theorem 2.2.6 iv)]. □

We are now able to prove the main result of this Section.

Theorem 2.6. *Let H be a quasi-bialgebra, $\mathfrak{B}$ a left H-comodule algebra and C a right H-module coalgebra. The categories ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ and $\mathrm{Rat}(\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}})$ are isomorphic, and $\mathrm{Rat}(\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}})$ is equal to $\sigma_{C^*\blacktriangleright\!\!<\mathfrak{B}}[C\otimes\mathfrak{B}]$.*

Proof. Recall that we have a functor

$$\mathfrak{G} : {}^C\mathcal{M}(H)_{\mathfrak{B}} \to \mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}},\ \mathfrak{G}(M) = M,$$

with right $C^*\blacktriangleright\!\!<\mathfrak{B}$-action given by the formula

$$m \cdot (c^*\blacktriangleright\!\!<\mathfrak{b}) = c^*(m_{\{-1\}})m_{\{0\}} \cdot \mathfrak{b}.$$

It is clear that $\mathfrak{G}(M)$ is rational as a $C^*\blacktriangleright\!\!<\mathfrak{B}$-module.
Consider a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module M. Then M is a right $\mathfrak{B}$-module through $m \cdot \mathfrak{b} = m \cdot (\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b})$. We define a linear map $\lambda_M :\ M \to C \otimes M$ as follows:

$$\lambda_M(m) = \sum_i c_i \otimes m_i$$

if and only if

$$m \cdot (c^*\blacktriangleright\!\!<\mathfrak{b}) = \sum_i c^*(c_i)m_i \cdot (\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}),$$

for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. It is clear that λ_M is well-defined.
Fix i, and assume that

$$\lambda_M(m_i) = \sum_j c^i_j \otimes m^i_j,$$

or, equivalently,

$$m_i \cdot (c^*\blacktriangleright\!\!<\mathfrak{b}) = c^*(c^i_j)m^i_j \cdot (\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}),$$

for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. Therefore we have, for all $c^*, d^* \in C^*$ and $m \in M$, that

$$\begin{aligned}
\langle c^* \otimes d^* \otimes id_M, (id_C \otimes \lambda_M)(\lambda_M(m)) \cdot \Phi_\lambda\rangle &= c^*(c_i \cdot \tilde{X}^1_\lambda)d^*(c^i_j \cdot \tilde{X}^2_\lambda)m^i_j \cdot \tilde{X}^3_\lambda \\
&= c^*(c_i \cdot \tilde{X}^1_\lambda)m_i \cdot (\tilde{X}^2 \cdot d^*\blacktriangleright\!\!<\tilde{X}^3_\lambda) = (m \cdot (\tilde{X}^1_\lambda \cdot c^*\blacktriangleright\!\!<1_{\mathfrak{B}})) \cdot (\tilde{X}^2_\lambda \cdot d^*\blacktriangleright\!\!<\tilde{X}^3_\lambda) \\
&= m \cdot ((\tilde{X}^1_\lambda \cdot c^*\blacktriangleright\!\!<1_{\mathfrak{B}})(\tilde{X}^2_\lambda \cdot d^*\blacktriangleright\!\!<\tilde{X}^3_\lambda)) = m \cdot (c^*d^*\blacktriangleright\!\!<1_{\mathfrak{B}}) \\
&= c^*d^*(c_i)m_i = \langle c^* \otimes d^* \otimes id_M, (\underline{\Delta} \otimes id_M)(\lambda_M(m))\rangle,
\end{aligned}$$

proving that (2.7) is satisfied. (2.8) is trivial since $m = m\cdot(\underline{\varepsilon}\blacktriangleright\!\!<1_{\mathfrak{B}})$ for all $m \in M$.
We will next prove that (2.9) holds. First observe that

$$\begin{aligned}
(m \cdot \mathfrak{b}) \cdot (c^*\blacktriangleright\!\!<\mathfrak{b}') &= (m \cdot (\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b})) \cdot (c^*\blacktriangleright\!\!<\mathfrak{b}') = m \cdot ((\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b})(c^*\blacktriangleright\!\!<\mathfrak{b}')) \\
&= \ \ m \cdot (\mathfrak{b}_{[-1]} \cdot c^*\blacktriangleright\!\!<\mathfrak{b}_{[0]}\mathfrak{b}') = c^*(c_i \cdot \mathfrak{b}_{[-1]})(m_i \cdot \mathfrak{b}_{[0]}) \cdot (\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}'),
\end{aligned}$$

for all $m \in M$, $c^* \in C^*$ and $\mathfrak{b}, \mathfrak{b}' \in \mathfrak{B}$. This shows that $\lambda_M(m\cdot\mathfrak{b}) = c_i\cdot\mathfrak{b}_{[-1]}\otimes m_i\cdot\mathfrak{b}_{[0]}$, as needed, and it follows that $M \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$.
Let $\eta :\ M \to N$ be a morphism of rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-modules. Take $m \in M$, and assume that $\lambda_M(m) = \sum\limits_i c_i \otimes m_i$. We compute that

$$\eta(m) \cdot (c^*\blacktriangleright\!\!<\mathfrak{b}) = \eta(m \cdot (c^*\blacktriangleright\!\!<\mathfrak{b})) = \sum_i c^*(c_i)\eta(m_i) \cdot (\underline{\varepsilon}\blacktriangleright\!\!<\mathfrak{b}),$$

for all $c^* \in C^*$ and $\mathfrak{b} \in \mathfrak{B}$. This is equivalent to $\lambda_N(\eta(m)) = c_i \otimes \eta(m_i) = (id_C \otimes \eta)(\lambda_M(m))$, hence η is left C-colinear. It is clear that η is right $\mathfrak{B}$-linear, so η is a morphism in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$, and we have a functor $\mathfrak{F} : \mathrm{Rat}(\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}}) \to {}^C\mathcal{M}(H)_{\mathfrak{B}}$, which is inverse to $\mathfrak{G}$.

The proof of the fact that $\mathrm{Rat}(\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}})$ and $\sigma_{C^*\blacktriangleright\!\!<\mathfrak{B}}[C \otimes \mathfrak{B}]$ are equal is similar to the proof of [19, Lemma 3.9].
We first show that $M \in \sigma_{C^*\blacktriangleright\!\!<\mathfrak{B}}[C \otimes \mathfrak{B}]$ for every $M \in \mathrm{Rat}(\mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}})$. We recall first that a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module belongs to $\sigma_{C^*\blacktriangleright\!\!<\mathfrak{B}}[C \otimes \mathfrak{B}]$ if and only if there

exists a set I, a right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module N, and two $C^*\blacktriangleright\!\!<\mathfrak{B}$-linear maps $\iota : M \to N$ and $\pi : (C\otimes\mathfrak{B})^{(I)} \to N$ such that ι is injective and π is surjective.
Let M be a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module. Thus $M \in {}^C\mathcal{M}(H)_{\mathfrak{B}}$ and $\mathfrak{G}(M) = M$ as right $C^*\blacktriangleright\!\!<\mathfrak{B}$-modules. It is easy to check that $\iota = \lambda_M : M \to C\otimes M$ is an injective morphism in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$; here $C\otimes M$ has the right-left Doi-Hopf module structure given by (2.10, 2.11). The map

$$\pi_M : \mathfrak{B}^{(M)} \to M,\ \pi_M((\mathfrak{b}_m)_m) = m\cdot\mathfrak{b}_m,$$

is a surjective right $\mathfrak{B}$-linear map. By Proposition 2.2, it provides a surjective morphism $id_C\otimes\pi_M : C\otimes\mathfrak{B}^{(M)} \to C\otimes M$ in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$, and since $C\otimes\mathfrak{B}^{(M)} \cong (C\otimes\mathfrak{B})^{(M)}$ in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$, we conclude that there exists a surjective morphism $\pi : (C\otimes\mathfrak{B})^{(M)} \to C\otimes M$ in ${}^C\mathcal{M}(H)_{\mathfrak{B}}$, so $M \in \sigma_{C^*\blacktriangleright\!<\mathfrak{B}}[C\otimes\mathfrak{B}]$.
Take $M \in \sigma_{C^*\blacktriangleright\!<\mathfrak{B}}[C\otimes\mathfrak{B}]$. Then we have right $C^*\blacktriangleright\!\!<\mathfrak{B}$-morphisms $\iota : M \to N$ and $\pi : (C\otimes\mathfrak{B})^{(I)} \to N$ such that ι is injective and π is surjective. Since the right $C^*\blacktriangleright\!\!<\mathfrak{B}$-module $C\otimes\mathfrak{B}$ lies in the image of $\mathfrak{G}$, it follows that $C\otimes\mathfrak{B}$ is a rational module. By Proposition 2.5 iii), $(C\otimes\mathfrak{B})^{(I)}$ is a rational module too and since π is surjective we deduce from Proposition 2.5 ii) that N is rational. Finally, from Proposition 2.5 ii) and the fact that ι is injective, it follows that M is a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module. □

Remark 2.7. It is well-known that the category $\sigma_{\mathcal{A}}[M]$ is a Grothendieck category, so it follows from Theorem 2.6 that ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is a Grothendieck category, an observation that we already made before.

Corollary 2.8. [3, Proposition 5.2] *Let H be a quasi-bialgebra, $\mathfrak{B}$ a left H-comodule algebra and C a finite dimensional right H-module coalgebra. Then the categories ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ and $\mathcal{M}_{C^*\blacktriangleright\!<\mathfrak{B}}$ are isomorphic.*

Corollary 2.9. *Let M be a right-left $(H,\mathfrak{B},C)$-Hopf module. Then the following assertions hold:*

i) *The right-left $(H,\mathfrak{B},C)$-Hopf submodule generated by an element of M is finite dimensional.*
ii) *M is the sum of its finite dimensional $(H,\mathfrak{B},C)$-Hopf submodules.*

Proof. i) M is a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module, so the right-left $(H,\mathfrak{B},C)$-Hopf submodule generated by an element $m\in M$ coincides with the cyclic $C^*\blacktriangleright\!\!<\mathfrak{B}$-submodule generated by m, by Theorem 2.6. We know from Proposition 2.5 i) that it is finite dimensional.
ii) View M as a rational $C^*\blacktriangleright\!\!<\mathfrak{B}$-module. Obviously, M is the sum of its cyclic $C^*\blacktriangleright\!\!<\mathfrak{B}$-submodules and all of these are finite dimensional right-left $(H,\mathfrak{B},C)$-Hopf submodules. □

2.4. **Doi-Hopf modules and Koppinen's smash product.** We begin this Section with some general results about corings, taken from [2, Sec. 19 and 20]. Let R be a ring, and $\mathcal{C}$ an R-coring. Then ${}^*\mathcal{C} = {}_R\mathrm{Hom}(\mathcal{C},R)$ is an R-ring, with multiplication

$$(\varphi\#\psi)(c) = \psi(c_{(1)}\varphi(c_{(2)})),$$

for all $\varphi,\psi \in {}^*\mathcal{C}$. We have a functor $F : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_{{}^*\mathcal{C}}$, $F(M) = M$, with $m\cdot\varphi = m_{[0]}\varphi(m_{[1]})$, for all $m\in M$ and $\varphi\in{}^*\mathcal{C}$. If $\mathcal{C}$ is finitely generated and projective as a left R-module, then F is an isomorphism of categories.

Assume now that $\mathcal{C}$ is locally projective as a left R-module. Then $\mathcal{M}^{\mathcal{C}}$ is isomorphic to the category $\sigma_{^*\mathcal{C}}[\mathcal{C}]$, see [2, 19.3]; observe that our multiplication on ${}^*\mathcal{C}$ is opposite to the one from [2], so that left ${}^*\mathcal{C}$-modules in [2] are our right ${}^*\mathcal{C}$-modules. Take a right ${}^*\mathcal{C}$-module M. $m \in M$ is called rational if there exists $\sum\limits_i m_i \otimes c_i \in M \otimes_R \mathcal{C}$ such that $m \cdot \varphi = \sum\limits_i m_i \varphi(c_i)$, for all $\varphi \in {}^*\mathcal{C}$. Then $\mathrm{Rat}^{\mathcal{C}}(M) = \{m \in M \mid m \text{ is rational}\}$ is a right $\mathcal{C}$-comodule, and we obtain a functor $\mathrm{Rat}^{\mathcal{C}} :\ \mathcal{M}_{^*\mathcal{C}} \to \mathcal{M}^{\mathcal{C}}$, which is right adjoint to F. M is called rational if $\mathrm{Rat}^{\mathcal{C}}(M) = M$. $\mathcal{M}^{\mathcal{C}}$ is isomorphic to the full category of $\mathcal{M}_{^*\mathcal{C}}$ consisting of rational right ${}^*\mathcal{C}$-modules.

Now let H be a quasi-bialgebra, $\mathfrak{B}$ a left H-comodule algebra, and C a right H-comodule algebra. We consider the $\mathfrak{B}$-coring $\mathcal{C} = \mathfrak{B} \otimes C$ from Section 2.2. Since we work over a field k, C is projective as a k-module, hence $\mathcal{C}$ is projective (and a fortiori locally projective) as a left $\mathfrak{B}$-module. Hence we can apply the above results to this situation. We have an isomorphism of vector spaces

$$ {}^*\mathcal{C} = {}_{\mathfrak{B}}\mathrm{Hom}(\mathfrak{B} \otimes C, \mathfrak{B}) \cong \mathrm{Hom}(C, \mathfrak{B}). $$

The multiplication on ${}^*\mathcal{C}$ can be transported to a multiplication on $\mathrm{Hom}(C, \mathfrak{B})$. This multiplication makes $\mathrm{Hom}(C, \mathfrak{B})$ into a B-ring $\#(C, \mathfrak{B})$, which we will call the Koppinen smash product. The multiplication is given by the following formula:

$$ (f\#g)(c) = \tilde{x}^3_\lambda f(c_{\underline{1}} \cdot \tilde{x}^1_\lambda)_{[0]} g\Big(c_{\underline{2}} \cdot \tilde{x}^2_\lambda f(c_{\underline{1}} \cdot \tilde{x}^1_\lambda)_{[1]}\Big). \tag{2.16} $$

In the situation where H is an associative bialgebra, we recover the smash product introduced first by Koppinen in [16]. The relation to the generalized smash product introduced in Section 2.3 is discussed in Proposition 2.10.

Proposition 2.10. *The k-linear map*

$$ \alpha :\ C^* \blacktriangleright\!\!< \mathfrak{B} \to \#(C, B),\ \ \alpha(c^* \blacktriangleright\!\!< \mathfrak{b}) = f, $$

with $f(c) = \langle c^*, c\rangle b$, for all $c \in C$, is a morphism of $\mathfrak{B}$-rings. It is an isomorphism if C is finite dimensional.

Proof. We have to show that α is multiplicative. Take $c^* \blacktriangleright\!\!< \mathfrak{b}, d^* \blacktriangleright\!\!< \mathfrak{b}' \in C^* \blacktriangleright\!\!< \mathfrak{B}$, and write $\alpha(c^* \blacktriangleright\!\!< \mathfrak{b}) = f$, $\alpha(d^* \blacktriangleright\!\!< \mathfrak{b}') = g$. Using (2.16), we compute that

$$ (f\#g)(c) = \tilde{x}^3_\lambda \langle c^*, c_{\underline{1}} \cdot \tilde{x}^1_\lambda\rangle \mathfrak{b}_{[0]} \langle d^*, c_{\underline{2}} \cdot \tilde{x}^2_\lambda \mathfrak{b}_{[1]}\rangle \mathfrak{b}'. $$

We also have that

$$ (c^* \blacktriangleright\!\!< \mathfrak{b})(d^* \blacktriangleright\!\!< \mathfrak{b}') = (\tilde{x}^1_\lambda \cdot c^*)(\tilde{x}^2_\lambda \mathfrak{b}_{[-1]} \cdot d^*) \blacktriangleright\!\!< \tilde{x}^3_\lambda \mathfrak{b}_{[0]} \mathfrak{b}', $$

so

$$ \alpha\big((c^* \blacktriangleright\!\!< \mathfrak{b})(d^* \blacktriangleright\!\!< \mathfrak{b}')\big)(c) = \langle c^*, c_{\underline{1}} \cdot \tilde{x}^1_\lambda\rangle \langle d^*, c_{\underline{2}} \cdot \tilde{x}^2_\lambda \mathfrak{b}_{[1]}\rangle \tilde{x}^3_\lambda \mathfrak{b}_{[0]} \mathfrak{b}', $$

as needed. □

Take a right $\#(C, B)$-module M. $m \in M$ is rational if there exists $\sum\limits_i m_i \otimes c_i \in M \otimes C$ such that

$$ m \cdot f = \sum_i m_i f(c_i), $$

for all $f :\ C \to B$. M is called rational if every $m \in M$ is rational.

Corollary 2.11. *Now let H be a quasi-bialgebra, $\mathfrak{B}$ a left H-comodule algebra, and C a right H-comodule algebra. Then the category ${}^C\mathcal{M}(H)_{\mathfrak{B}}$ is isomorphic to the full subcategory of $\mathcal{M}_{\#(C,B)}$, which is also equal to $\sigma_{\#(C,\mathfrak{B})}(\mathfrak{B}\otimes C)$.*

2.5. **Left, right and right-left Doi-Hopf modules.** For the sake of completeness, we also define the other Doi-Hopf module categories. In fact, we have four different types of Doi-Hopf modules. The first one was already studied, namely the right-left version. We also have the left-right, right-right and left-left versions.

Definition 2.12. Let H be a quasi-bialgebra, $\mathfrak{A}$ a right H-comodule algebra and $\mathfrak{B}$ a left H-comodule algebra. In the statements below we assume in i) and iii) that C is a left H-module coalgebra, and in ii) that C is a right H-module coalgebra, respectively.

i) A left-right $(H,\mathfrak{A},C)$-Hopf module (or Doi-Hopf module) is a left $\mathfrak{A}$-module M together with a k-linear map $\rho_M:\ M\to M\otimes C$, $\rho_M(m)=m_{(0)}\otimes m_{(1)}$, such that the following relations hold, for all $m\in M$ and $\mathfrak{a}\in\mathfrak{A}$:

$$\Phi_\rho\cdot(\rho_M\otimes id_C)(\rho_M(m))=(id_M\otimes\underline{\Delta})(\rho_M(m)), \tag{2.17}$$

$$(id_M\otimes\underline{\varepsilon})(\rho_M(m))=m, \tag{2.18}$$

$$\rho_M(\mathfrak{a}\cdot m)=\mathfrak{a}_{\langle 0\rangle}\cdot m_{(0)}\otimes\mathfrak{a}_{\langle 1\rangle}\cdot m_{(1)}. \tag{2.19}$$

${}_{\mathfrak{A}}\mathcal{M}(H)^C$ is the category of left-right $(H,\mathfrak{A},C)$-Hopf modules and left $\mathfrak{A}$-linear, right C-colinear maps.

ii) A right-right $(H,\mathfrak{A},C)$-Hopf module (or Doi-Hopf module) is a right $\mathfrak{A}$-module M together with a k-linear map $\rho_M:\ M\to M\otimes C$, $\rho_M(m)=m_{(0)}\otimes m_{(1)}$, such that the following relations hold, for all $m\in M$ and $\mathfrak{a}\in\mathfrak{A}$:

$$(\rho_M\otimes id_M)(\rho_M(m))=(id_M\otimes\underline{\Delta})(\rho_M(m))\cdot\Phi_\rho, \tag{2.20}$$

$$(id_M\otimes\underline{\varepsilon})(\rho_M(m))=m, \tag{2.21}$$

$$\rho_M(m\cdot\mathfrak{a})=m_{(0)}\cdot\mathfrak{a}_{\langle 0\rangle}\otimes m_{(1)}\cdot\mathfrak{a}_{\langle 1\rangle}. \tag{2.22}$$

$\mathcal{M}(H)^C_{\mathfrak{A}}$ is the category of right-right $(H,\mathfrak{A},C)$-Hopf modules and left $\mathfrak{A}$-linear, right C-colinear maps.

iii) A left-left $(H,\mathfrak{B},C)$-Hopf module (or Doi-Hopf module) is a left $\mathfrak{B}$-module M together with a left k-linear map $\lambda_M:\ M\to C\otimes M$, $\lambda_M(m)=m_{\{-1\}}\otimes m_{\{0\}}$, such that the following relations hold, for all $m\in M$ and $\mathfrak{b}\in\mathfrak{B}$:

$$\Phi_\lambda\cdot(\underline{\Delta}\otimes id_M)(\lambda_M(m))=(id_C\otimes\lambda_M)(\lambda_M(m)), \tag{2.23}$$

$$(\underline{\varepsilon}\otimes id_M)(\lambda_M(m))=m, \tag{2.24}$$

$$\lambda_M(\mathfrak{b}\cdot m)=\mathfrak{b}_{[-1]}\cdot m_{\{-1\}}\otimes\mathfrak{b}_{[0]}\cdot m_{\{0\}}. \tag{2.25}$$

${}^C_{\mathfrak{B}}\mathcal{M}(H)$ is the category of left-left $(H,\mathfrak{A},C)$-Hopf modules and left $\mathfrak{A}$-linear,left C-colinear maps.

We remind that if $(\mathfrak{A},\rho,\Phi_\rho)$ is a right H-comodule algebra then $\underline{\mathfrak{A}}^{\rm op}=(\mathfrak{A}^{\rm op},\rho\circ\tau_{\mathfrak{A},H},(\Phi_\rho)^{321})$ is a left $H^{\rm op,cop}$-comodule algebra and $\underline{\mathfrak{A}}=(\mathfrak{A},\rho\circ\tau_{\mathfrak{A},H},(\Phi_\rho^{-1})^{321})$ is a left $H^{\rm cop}$-comodule algebra, where $\tau_{\mathfrak{A},H}:\ \mathfrak{A}\otimes H\to H\otimes\mathfrak{A}$ is the switch map. Also, if $\mathfrak{B}$ is a left H-comodule algebra then $\mathfrak{B}^{\rm op}=(\mathfrak{B}^{\rm op},\lambda,\Phi_\lambda^{-1})$ is a left $H^{\rm op}$-comodule algebra.

On the other hand, if C is a left H-module coalgebra then C is a right $H^{\rm op}$-module coalgebra and $C^{\rm cop}$ is a right $H^{\rm op,cop}$-module coalgebra (and vice versa). So if C is a right H-module coalgebra then $C^{\rm cop}$ is a right $H^{\rm cop}$-module coalgebra.
Having these correspondences one can easily see that

$$_{\mathfrak{A}}\mathcal{M}(H)^C \cong {}^{C^{\rm cop}}\mathcal{M}(H^{\rm op,cop})_{\underline{\mathfrak{A}}^{\rm op}},\ \mathcal{M}(H)^C_{\mathfrak{A}} \cong {}^{C^{\rm cop}}\mathcal{M}(H^{\rm cop})_{\underline{\mathfrak{A}}},$$

$$^C_{\mathfrak{B}}\mathcal{M}(H) \cong {}^C\mathcal{M}(H^{\rm op})_{\mathfrak{B}^{\rm op}}.$$

It follows that the four different types of Doi-Hopf modules are isomorphic to categories of comodules over suitable corings, and they are Grothendieck categories with enough injective objects. On the other hand, if C is finite dimensional then the above categories are isomorphic to categories of modules over certain generalized smash product algebras. More precisely, we have:

Remarks 2.13. i) The category $_{\mathfrak{A}}\mathcal{M}(H)^C$ is isomorphic to the category of right modules over the generalized smash product $(C^{\rm cop})^*\blacktriangleright\!\!<\underline{\mathfrak{A}}^{\rm op}$ (over $H^{\rm op,cop}$), and therefore also to the category of left modules over $((C^{\rm cop})^*\blacktriangleright\!\!<\underline{\mathfrak{A}}^{\rm op})^{\rm op}$. Is not hard to see that the multiplication rule in $((C^{\rm cop})^*\blacktriangleright\!\!<\underline{\mathfrak{A}}^{\rm op})^{\rm op}$ is

$$(c^*\blacktriangleright\!\!<\mathfrak{a})(d^*\blacktriangleright\!\!<\mathfrak{a}') = (c^*\cdot \mathfrak{a}'_{\langle 1\rangle}\tilde{x}^2_\rho)(d^*\cdot\tilde{x}^3_\rho)\blacktriangleright\!\!<\mathfrak{a}\mathfrak{a}'_{\langle 0\rangle}\tilde{x}^1_\rho,$$

for all $c^*, d^*\in C^*$ and $\mathfrak{a}, \mathfrak{a}'\in\mathfrak{A}$, where $(c^*\cdot h)(c) = c^*(h\cdot c)$ for all $c^*\in C^*$, $h\in H$ and $c\in C$. Therefore, under the trivial permutation of tensor factors we have that $((C^{\rm cop})^*\blacktriangleright\!\!<\underline{\mathfrak{A}}^{\rm op})^{\rm op} = \mathfrak{A}>\!\!\blacktriangleleft C^*$, the right generalized smash product between the right H-comodule algebra $\mathfrak{A}$ and the right H-module algebra C^* (see [7] for more details). We conclude that $_{\mathfrak{A}}\mathcal{M}(H)^C \cong {}_{\mathfrak{A}>\!\!\blacktriangleleft C^*}\mathcal{M}$ if C is finite dimensional.

ii) The above arguments entail that

$$\mathcal{M}(H)^C_{\mathfrak{A}} \cong {}^{C^{\rm cop}}\mathcal{M}(H^{\rm op})_{\underline{\mathfrak{A}}} \cong \mathcal{M}_{(C^{\rm cop})^*\blacktriangleright\!\!<\underline{\mathfrak{A}}} \cong {}_{((C^{\rm cop})^*\blacktriangleright\!\!<\underline{\mathfrak{A}})^{\rm op}}\mathcal{M},$$

where the generalized smash product is over $H^{\rm cop}$. The explicit formula for the multiplication $\odot$ on $((C^{\rm cop})^*\blacktriangleright\!\!<\underline{\mathfrak{A}})^{\rm op}$ is given by

$$(c^*\blacktriangleright\!\!<\mathfrak{a})\odot(d^*\blacktriangleright\!\!<\mathfrak{a}') = (\tilde{X}^2_\rho\mathfrak{a}'_{\langle 1\rangle}\cdot c^*)(\tilde{X}^3_\rho\cdot d^*)\blacktriangleright\!\!<\tilde{X}^1_\rho\mathfrak{a}'_{\langle 0\rangle}\mathfrak{a}, \tag{2.26}$$

for all $c^*, d^*\in C^*$ and $\mathfrak{a}, \mathfrak{a}'\in\mathfrak{A}$.
iii) Obviously, $^C_{\mathfrak{B}}\mathcal{M}(H) \cong \mathcal{M}_{C^*\blacktriangleright\!\!<\mathfrak{B}^{\rm op}}$, where the generalized smash product is taken over $H^{\rm op}$.

Let H be a quasi-Hopf algebra and C a finite dimensional right H-module coalgebra. It was proved in [6, Proposition 3.2] that the category $\mathcal{M}^C_H = \mathcal{M}(H)^C_H$ is isomorphic to the category of left modules over the smash product algebra $C^*\#H = C^*\blacktriangleright\!\!<H$. Now, by Remark 2.13 ii) the category $\mathcal{M}^C_H$ is isomorphic to the category of left modules over $((C^{\rm cop})^*\#\underline{H})^{\rm op}$. The next result shows that the smash product algebras $C^*\#H$ and $((C^{\rm cop})^*\#\underline{H})^{\rm op}$ are isomorphic.

Proposition 2.14. *Let H be a quasi-Hopf algebra and C a right H-module coalgebra. Then the map*

$$\varphi:\ C^*\#H\to((C^{\rm cop})^*\#\underline{H})^{\rm op},\ \varphi(c^*\#h) = S^{-1}(\mathfrak{q}^1h_1g^1)\cdot c^*\#S^{-1}(\mathfrak{q}^2h_2g^2)$$

is an algebra isomorphism. Here $f^{-1} = g^1\otimes g^2$ is the element defined by (1.12) and $\tilde{q}_\Delta = \mathfrak{q}^1\otimes\mathfrak{q}^2$ is the element $\tilde{q}_\lambda$ defined in (1.25), in the special case where $\mathfrak{B} = H$.

Proof. We first show that φ is an algebra map. Let $f = f^1 \otimes f^2$, $f^{-1} = g^1 \otimes g^2 = G^1 \otimes G^2$ and $\tilde{q}_\Delta = \mathfrak{q}^1 \otimes \mathfrak{q}^2 = \mathfrak{Q}^1 \otimes \mathfrak{Q}^2$ be the elements defined by (1.11), (1.12) and (1.25), respectively. We compute

$$\begin{aligned}
\varphi((c^*\#h)(d^*\#h')) &= \varphi((x^1\cdot c^*)(x^2h_1\cdot d^*)\#x^3h_2h') \\
&\overset{(2.13,1.10)}{=} [S^{-1}(f^2\mathfrak{q}^1_2x^3_{(1,2)}(h_2h')_{(1,2)}g^1_2G^2)x^1\cdot c^*] \\
&\qquad [S^{-1}(f^1\mathfrak{q}^1_1x^3_{(1,1)}(h_2h')_{(1,1)}g^1_1G^1)x^2h_1\cdot d^*]\#S^{-1}(\mathfrak{q}^2x^3_2(h_2h')_2g^2) \\
&\overset{(1.31,1.1)}{=} [S^{-1}(\mathfrak{q}^1\mathfrak{Q}^2_1(h_2h')_{(2,1)}X^2g^1_2G^2)\cdot c^*][S^{-1}(\mathfrak{Q}^1(h_2h')_1X^1g^1_1G^1)h_1\cdot d^*] \\
&\qquad \#S^{-1}(\mathfrak{q}^2\mathfrak{Q}^2_2(h_2h')_{(2,2)}X^3g^2)
\end{aligned}$$

$$\begin{aligned}
&\overset{(1.27,1.8,1.13)}{=} [X^2S^{-1}(\mathfrak{q}^1h_1\mathfrak{Q}^2_1h'_{(2,1)}g^2_1G^1)\cdot c^*][X^3S^{-1}(\mathfrak{Q}^1h'_1g^1)\cdot d^*] \\
&\qquad \#X^1S^{-1}(\mathfrak{q}^2h_2\mathfrak{Q}^2_2h'_{(2,2)}g^2_2G^2) \\
&\overset{(1.10)}{=} [X^2S^{-1}(\mathfrak{Q}^2h'_2g^2)_2S^{-1}(\mathfrak{q}^1h_1G^1)\cdot c^*][X^3S^{-1}(\mathfrak{Q}^1h'_1g^1)\cdot d^*] \\
&\qquad \#X^1S^{-1}(\mathfrak{Q}^2h'_2g^2)_1S^{-1}(\mathfrak{q}^2h_2G^2) \\
&\overset{(2.26)}{=} \varphi(c^*\#h)\odot\varphi(d^*\#h'),
\end{aligned}$$

as needed. It is clear that $\varphi(\underline{\varepsilon}\#1_H) = \underline{\varepsilon}\#1_H$, so it remains to be shown that φ is bijective.
First we introduce some notation. Let $\mathfrak{A}$ be a right H-comodule algebra, and define the element $\tilde{q}_\rho \in \mathfrak{A}\otimes H$ as follows:

$$\tilde{q}_\rho = \tilde{q}^1_\rho \otimes \tilde{q}^2_\rho = \tilde{X}^1_\rho \otimes S^{-1}(\alpha\tilde{X}^3_\rho)\tilde{X}^2_\rho. \tag{2.27}$$

In the special situation where $\mathfrak{A} = H$, the element $\tilde{q}_\Delta$ will be denoted by $\tilde{q}_\Delta = q^1\otimes q^2$.
We now claim that $\varphi^{-1} :\ ((C^{\rm cop})^*\#\underline{H})^{\rm op} \to C^*\#H$ is given by the formula

$$\varphi^{-1}(c^*\#h) = g^1S(q^2h_2)\cdot c^*\#g^2S(q^1h_1),$$

for all $c^*\in C^*$ and $h\in H$.
From [6, Lemma 2.6], we recall the following formula:

$$g^2\alpha S^{-1}(g^1) = S^{-1}(\beta). \tag{2.28}$$

Let $\tilde{p}_\Delta = \mathfrak{p}^1\otimes\mathfrak{p}^2$ be the element $\tilde{p}_\lambda$ defined in (1.25), in the special case where $\mathfrak{B} = H$. We then compute, for all $c^*\in C^*$ and $h\in H$:

$$\begin{aligned}
\varphi^{-1}\circ\varphi(c^*\#h) &= \varphi^{-1}(S^{-1}(\mathfrak{q}^1h_1g^1)\cdot c^*\#S^{-1}(\mathfrak{q}^2h_2g^2)) \\
&\overset{(1.10)}{=} \mathfrak{q}^2_1h_{(2,1)}g^2_1G^1S(q^2)S^{-1}(\mathfrak{q}^1h_1g^1)\cdot c^*\#\mathfrak{q}^2_2h_{(2,2)}g^2_2G^2S(q^1) \\
&\overset{(2.27,1.8,1.13)}{=} \mathfrak{q}^2_1h_{(2,1)}X^2g^1_2G^2\alpha S^{-1}(\mathfrak{q}^1h_1X^1g^1_1G^1)\cdot c^*\#\mathfrak{q}^2_2h_{(2,2)}X^3g^2 \\
&\overset{(2.28,1.5,1.25)}{=} S^{-1}(\mathfrak{q}^1h_1S(\mathfrak{q}^2_1h_{(2,1)}\mathfrak{p}^1))\cdot c^*\#\mathfrak{q}^2_2h_{(2,2)}\mathfrak{p}^2 \\
&\overset{(1.26,1.28)}{=} c^*\#h.
\end{aligned}$$

The proof of the fact that $\varphi\circ\varphi^{-1} = id_{((C^{\rm cop})^*\#\underline{H})^{\rm op}}$ is based on similar computations. $\square$

Remark 2.15. The isomorphism φ in Proposition 2.14 can be defined more generally for a left H-module algebra A instead of a right H-module coalgebra C. Observe that H cannot be replaced by an H-bicomodule algebra $\mathbb{A}$, because of the appearance of the antipode S of H on the second position of the tensor product.

3. Yetter-Drinfeld modules are Doi-Hopf modules

In this Section, we will show that Yetter-Drinfeld modules are special case of Doi-Hopf modules. We will then apply the properties of Doi-Hopf modules to Yetter-Drinfeld modules.

3.1. **Yetter-Drinfeld modules over quasi-bialgebras.** The category of Yetter-Drinfeld modules over a quasi-Hopf algebra H was introduced by Majid, as the center of the monoidal category ${}_H\mathcal{M}$. His aim was to define the quantum double by an implicit Tannaka-Krein reconstruction procedure, see [17]. From [7], we recall the following more general definition of Yetter-Drinfeld modules.
The category of (H,H)-bimodules, ${}_H\mathcal{M}_H$, is monoidal. The associativity constraints $\mathbf{a}'_{U,V,W} : (U\otimes V)\otimes W\to U\otimes(V\otimes W)$ are given by

$$\mathbf{a}'_{U,V,W}((u\otimes v)\otimes w)=\Phi\cdot(u\otimes(v\otimes w))\cdot\Phi^{-1} \tag{3.1}$$

for all $U,V,W\in{}_H\mathcal{M}_H$, $u\in U$, $v\in V$ and $w\in W$. A coalgebra in the category of (H,H)-bimodules will be called an H-bimodule coalgebra. More precisely, an H-bimodule coalgebra C is an (H,H)-bimodule (denote the actions by $h\cdot c$ and $c\cdot h$) with a comultiplication $\underline{\Delta}:\ C\to C\otimes C$ and a counit $\underline{\varepsilon}:\ C\to k$ satisfying the following relations, for all $c\in C$ and $h\in H$:

$$\Phi\cdot(\underline{\Delta}\otimes id_C)(\underline{\Delta}(c))\cdot\Phi^{-1}=(id_C\otimes\underline{\Delta})(\underline{\Delta}(c)), \tag{3.2}$$

$$\underline{\Delta}(h\cdot c)=h_1\cdot c_{\underline{1}}\otimes h_2\cdot c_{\underline{2}},\ \underline{\Delta}(c\cdot h)=c_{\underline{1}}\cdot h_1\otimes c_{\underline{2}}\cdot h_2, \tag{3.3}$$

$$(\underline{\varepsilon}\otimes id_C)\circ\underline{\Delta}=(id_C\otimes\underline{\varepsilon})\circ\underline{\Delta}=id_C, \tag{3.4}$$

$$\underline{\varepsilon}(h\cdot c)=\varepsilon(h)\underline{\varepsilon}(c),\ \underline{\varepsilon}(c\cdot h)=\underline{\varepsilon}(c)\varepsilon(h), \tag{3.5}$$

where we used the same Sweedler-type notation as introduced before.
For further use we note that an H-bimodule coalgebra C can be always viewed as a left $H^{\rm op}\otimes H$-module coalgebra via the left $H^{\rm op}\otimes H$-action given for all $c\in C$ and $h,h'\in H$ by

$$(h\otimes h')\cdot c=h'\cdot c\cdot h. \tag{3.6}$$

Definition 3.1. ([7]) Let H be a quasi-bialgebra, C an H-bimodule coalgebra and $\mathbb{A}$ an H-bicomodule algebra. A left-right $(H,\mathbb{A},C)$-Yetter-Drinfeld module is a k-vector space M with the following additional structure:

- M is a left $\mathbb{A}$-module; we write $\cdot$ for the left $\mathbb{A}$-action;
- we have a k-linear map $\rho_M : M\to M\otimes C$, $\rho_M(m)=m_{(0)}\otimes m_{(1)}$, called the right C-coaction on M, such that for all $m\in M$, $\underline{\varepsilon}(m_{(1)})m_{(0)}=m$ and

$$\begin{aligned}&(\theta^2\cdot m_{(0)})_{(0)}\otimes(\theta^2\cdot m_{(0)})_{(1)}\cdot\theta^1\otimes\theta^3\cdot m_{(1)}\\ &=\tilde{x}^1_\rho\cdot(\tilde{x}^3_\lambda\cdot m)_{(0)}\otimes\tilde{x}^2_\rho\cdot(\tilde{x}^3_\lambda\cdot m)_{(1)_{\underline{1}}}\cdot\tilde{x}^1_\lambda\otimes\tilde{x}^3_\rho\cdot(\tilde{x}^3_\lambda\cdot m)_{(1)_{\underline{2}}}\cdot\tilde{x}^2_\lambda,\end{aligned} \tag{3.7}$$

- for all $u\in\mathbb{A}$ and $m\in M$ the following compatibility relation holds:

$$u_{<0>}\cdot m_{(0)}\otimes u_{<1>}\cdot m_{(1)}=(u_{[0]}\cdot m)_{(0)}\otimes(u_{[0]}\cdot m)_{(1)}\cdot u_{[-1]}. \tag{3.8}$$

${}_{\mathbb{A}}\mathcal{YD}(H)^C$ will be the category of left-right $(H,\mathbb{A},C)$-Yetter-Drinfeld modules and maps preserving the $\mathbb{A}$-action C-coaction.

We have seen in Section 1.2 that H is an H-bicomodule algebra; it is also clear that H is an H-bimodule coalgebra (take $\underline{\Delta} = \Delta$ and $\underline{\varepsilon} = \varepsilon$). If we take $H = \mathbb{A} = C$ in Definition 3.1, then we recover the category of Yetter-Drinfeld modules introduced by Majid in [17], and studied in [4, 5]. It is remarkable that a quasi-bialgebra H is a coalgebra in the category of H-bimodules, but not in the category of vector spaces, or in the category of left (or right) H-modules.

Let H be a quasi-bialgebra, $\mathbb{A}$ an H-bicomodule algebra and C a left H-module coalgebra. It is straightforward to check that C with the right H-module structure given by ε is an H-bimodule coalgebra. Then (3.7) and (3.8) reduce respectively to (2.17) and (2.19), in which only the right H-coaction on $\mathbb{A}$ appears. So in this particular case the category ${}_{\mathbb{A}}\mathcal{M}(H)^C$ is just ${}_{\mathbb{A}}\mathcal{YD}(H)^C$.

In order to show that the Yetter-Drinfeld modules are special case of Doi-Hopf modules we need a Doi-Hopf datum. As we have already seen, an H-bimodule coalgebra C can be viewed as a left $H^{\mathrm{op}} \otimes H$-module via the structure defined in (3.6). In order to provide a right $H^{\mathrm{op}} \otimes H$-comodule algebra structure on an H-bicomodule algebra $\mathbb{A}$, we need the following result.

Lemma 3.2. *Let H be a quasi-Hopf algebra and $(\mathfrak{B}, \lambda, \Phi_\lambda)$ a left H-comodule algebra. Then $\mathfrak{B}$ is a right H^{op}-comodule algebra via the structure*

$$\rho: \ \mathfrak{B} \to \mathfrak{B} \otimes H, \ \ \rho(\mathfrak{b}) = \mathfrak{b}_{[0]} \otimes S^{-1}(\mathfrak{b}_{[-1]}), \tag{3.9}$$

$$\Phi_\rho = \tilde{x}^3_\lambda \otimes S^{-1}(f^2\tilde{x}^2_\lambda) \otimes S^{-1}(f^1\tilde{x}^1_\lambda) \in \mathfrak{B} \otimes H \otimes H, \tag{3.10}$$

where $f^1 \otimes f^2$ is the Drinfeld twist defined in (1.11). Moreover, if $(\mathfrak{B}, \lambda, \Phi_\lambda)$ and $(\mathfrak{B}, \lambda', \Phi_{\lambda'})$ are twist equivalent left H-comodule algebras then the corresponding right H^{op}-comodule algebras are also twist equivalent.

Proof. The relation (1.14) follows easily by applying (1.18) and (1.10), and the relations (1.16, 1.17) are trivial. We prove now (1.15). Since $\Phi_{\mathrm{op}} = \Phi^{-1}$ we have

$$\begin{aligned}
&\tilde{X}^1_\rho\tilde{Y}^1_\rho \otimes x^1 \cdot_{\mathrm{op}} (\tilde{X}^2_\rho)_1 \cdot_{\mathrm{op}} \tilde{Y}^3_\rho \otimes x^2 \cdot_{\mathrm{op}} (\tilde{X}^2_\rho)_2 \cdot_{\mathrm{op}} \tilde{Y}^3_\rho \otimes x^3 \cdot_{\mathrm{op}} \tilde{X}^3_\rho \\
&= \tilde{x}^3_\lambda\tilde{y}^3_\lambda \otimes S^{-1}(F^2\tilde{y}^2_\lambda)S^{-1}(f^2\tilde{x}^2_\lambda)_1 x^1 \\
&\qquad \otimes S^{-1}(F^1\tilde{y}^1_\lambda)S^{-1}(f^2\tilde{x}^2_\lambda)_2 x^2 \otimes S^{-1}(f^1\tilde{x}^1_\lambda)x^3 \\
&\overset{(1.10)}{=} \tilde{x}^3_\lambda\tilde{y}^3_\lambda \otimes S^{-1}(S(x^1)F^2f^2_2(\tilde{x}^2_\lambda)_2\tilde{y}^2_\lambda) \\
&\qquad \otimes S^{-1}(S(x^2)F^1f^2_1(\tilde{x}^2_\lambda)_1\tilde{y}^1_\lambda) \otimes S^{-1}(S(x^3)f^1\tilde{x}^1_\lambda) \\
&\overset{(1.8,1.13)}{=} \tilde{x}^3_\lambda\tilde{y}^3_\lambda \otimes S^{-1}(F^2x^3(\tilde{x}^2_\lambda)_2\tilde{y}^2_\lambda) \\
&\qquad \otimes S^{-1}(f^2F^1_2x^2(\tilde{x}^2_\lambda)_1\tilde{y}^1_\lambda) \otimes S^{-1}(f^1F^1_1x^1\tilde{x}^1_\lambda) \\
&\overset{(1.19,1.10)}{=} \tilde{y}^3_\lambda(\tilde{x}^3_\lambda)_{[0]} \otimes S^{-1}(F^2\tilde{y}^2_\lambda) \cdot_{\mathrm{op}} S^{-1}((\tilde{x}^3_\lambda)_{[-1]}) \\
&\qquad \otimes S^{-1}(F^1\tilde{y}^1_\lambda)_1 \cdot_{\mathrm{op}} S^{-1}(f^2\tilde{x}^2_\lambda) \otimes S^{-1}(F^1\tilde{y}^1_\lambda)_2 \cdot_{\mathrm{op}} S^{-1}(f^1\tilde{x}^1_\lambda) \\
&\overset{(3.9,3.10)}{=} \tilde{X}^1_\rho(\tilde{Y}^1_\rho)_{\langle 0\rangle} \otimes \tilde{X}^2_\rho \cdot_{\mathrm{op}} (\tilde{Y}^1_\rho)_{\langle 1\rangle} \otimes (\tilde{X}^3_\rho)_1 \cdot_{\mathrm{op}} \tilde{Y}^2_\rho \otimes (\tilde{X}^3_\rho)_2 \cdot_{\mathrm{op}} \tilde{Y}^3_\rho,
\end{aligned}$$

where we denote by $\cdot_{\mathrm{op}}$ the multiplication of H^{op} and by $F^1 \otimes F^2$ another copy of the Drinfeld twist f.

Finally, it is not hard to see that if the invertible element $\mathbb{U} = \mathbb{U}^1 \otimes \mathbb{U}^2 \in H \otimes \mathfrak{B}$ provides a twist equivalence between the left H-comodule algebras $(\mathfrak{B}, \lambda, \Phi_\lambda)$ and $(\mathfrak{B}, \lambda', \Phi_{\lambda'})$ then the invertible element $\mathbb{V} = \mathbb{U}^2 \otimes S^{-1}(\mathbb{U}^1) \in \mathfrak{B} \otimes H$ provides a twist equivalence between the associated right H^{op}-comodule algebras $(\mathfrak{B}, \rho, \Phi_\rho)$ and $(\mathfrak{B}, \rho', \Phi_{\rho'})$, respectively. □

Proposition 3.3. *Let H be a quasi-Hopf algebra and $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ an H-bicomodule algebra. We define two right $H^{\rm op} \otimes H$-coactions*

$$\rho_1, \rho_2 : \ \mathbb{A} \to \mathbb{A} \otimes (H^{\rm op} \otimes H)$$

on $\mathbb{A}$, and corresponding elements $\Phi_{\rho_1}, \Phi_{\rho_2} \in \mathbb{A} \otimes (H^{\rm op} \otimes H) \otimes (H^{\rm op} \otimes H)$ as follows:

$$\rho_1(u) = u_{\langle 0\rangle_{[0]}} \otimes \Big(S^{-1}(u_{\langle 0\rangle_{[-1]}}) \otimes u_{\langle 1\rangle}\Big), \tag{3.11}$$

$$\begin{aligned}\Phi_{\rho_1} = (\tilde{X}^1_\rho)_{[0]}\tilde{x}^3_\lambda \theta^2_{[0]} &\otimes \Big(S^{-1}(f^2(\tilde{X}^1_\rho)_{[-1]_2}\tilde{x}^2_\lambda\theta^2_{[-1]}) \otimes \tilde{X}^2_\rho\theta^3\Big)\\ &\otimes \Big(S^{-1}(f^1(\tilde{X}^1_\rho)_{[-1]_1}\tilde{x}^1_\lambda\theta^1) \otimes \tilde{X}^3_\rho\Big),\end{aligned} \tag{3.12}$$

and

$$\rho_2(u) = u_{[0]_{\langle 0\rangle}} \otimes \Big(S^{-1}(u_{[-1]}) \otimes u_{[0]_{\langle 1\rangle}}\Big), \tag{3.13}$$

$$\begin{aligned}\Phi_{\rho_2} = (\tilde{x}^3_\lambda)_{\langle 0\rangle}\tilde{X}^1_\rho\Theta^2_{\langle 0\rangle} &\otimes \Big(S^{-1}(f^2\tilde{x}^2_\lambda\Theta^1) \otimes (\tilde{x}^3_\lambda)_{\langle 1\rangle_1}\tilde{X}^2_\rho\Theta^2_{\langle 1\rangle}\Big)\\ &\otimes \Big(S^{-1}(f^1\tilde{x}^1_\lambda) \otimes (\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\Theta^3\Big).\end{aligned} \tag{3.14}$$

Then $(\mathbb{A}, \rho_1, \Phi_{\rho_1})$ and $(\mathbb{A}, \rho_2, \Phi_{\rho_2})$ are twist equivalent right $H^{\rm op} \otimes H$-comodule algebras.

Proof. The statement follows from Lemma 3.2. Indeed, we have seen at the end of Section 1 that $\mathbb{A}$ has two twist equivalent left $H \otimes H^{\rm op}$-comodule algebra structures. Identifying $(H \otimes H^{\rm op})^{\rm op}$ and $H^{\rm op} \otimes H$, and computing the induced right coactions we obtain the structures defined in (3.11-3.14). We point out that the reassociator, the antipode and the Drinfeld twist corresponding to $H \otimes H^{\rm op}$ are given by

$$\begin{aligned}&\Phi_{H\otimes H^{\rm op}} = (X^1 \otimes x^1) \otimes (X^2 \otimes x^2) \otimes (X^3 \otimes x^3),\\ &S_{H\otimes H^{\rm op}} = S \otimes S^{-1}, \ f_{H\otimes H^{\rm op}} = (f^1 \otimes S^{-1}(g^2)) \otimes (f^2 \otimes S^{-1}(g^1)),\end{aligned}$$

where, as usual, $g^1 \otimes g^2$ is the inverse of $f = f^1 \otimes f^2$. □

Let H be a quasi-Hopf algebra, C an H-bimodule coalgebra and $\mathbb{A}$ an H-bicomodule algebra. In the sequel, $\mathbb{A}^1$ and $\mathbb{A}^2$ will be our notation for the right $H^{\rm op} \otimes H$-comodule algebras $(\mathbb{A}, \rho_1, \Phi_{\rho_1})$ and $(\mathbb{A}, \rho_2, \Phi_{\rho_2})$. By the above arguments, it make sense to consider the left-right Doi-Hopf module categories ${}_{\mathbb{A}^1}\mathcal{M}(H^{\rm op} \otimes H)^C$ and ${}_{\mathbb{A}^2}\mathcal{M}(H^{\rm op} \otimes H)^C$. It will follow from Proposition 3.4 that these two categories are isomorphic.

Proposition 3.4. *Let H be a quasi-bialgebra, C a left H-module coalgebra and $\mathfrak{A}^1 = (\mathfrak{A}, \rho, \Phi_\rho)$ and $\mathfrak{A}^2 = (\mathfrak{A}, \rho', \Phi_{\rho'})$ two twist equivalent right H-comodule algebras. Then the categories ${}_{\mathfrak{A}^1}\mathcal{M}(H)^C$ and ${}_{\mathfrak{A}^2}\mathcal{M}(H)^C$ are isomorphic.*

Proof. If $\mathfrak{A}^1$ and $\mathfrak{A}^2$ are twist equivalent, then there exists $\mathbb{V} \in \mathfrak{A} \otimes H$ satisfying (1.22-1.24). Take $M \in {}_{\mathfrak{A}^1}\mathcal{M}(H)^C$; M becomes an object in ${}_{\mathfrak{A}^2}\mathcal{M}(H)^C$ by keeping the same left $\mathfrak{A}$-module structure and defining

$$\rho'_M : \ M \to M \otimes C, \ \rho'(m) = \mathbb{V} \cdot \rho_M(m).$$

Conversely, take $M \in {}_{\mathfrak{A}^2}\mathcal{M}(H)^C$ via the structures $\cdot$ and ρ'_M. Then M can be viewed as a left-right $(H, \mathfrak{A}^1, C)$-Hopf module via the same left $\mathfrak{A}$-action $\cdot$ and the right C-coaction ρ_M defined by

$$\rho_M : \ M \to M \otimes C, \ \rho_M(m) = \mathbb{V}^{-1} \cdot \rho'_M(m).$$

These correspondences define two functors which act as the identity on morphisms and produce inverse isomorphisms. $\square$

Remark 3.5. Let H be a quasi-bialgebra, and $F \in H \otimes H$ a gauge transformation. We can consider the twisted quasi-bialgebra H_F (see (1.7-1.8)), and it is well-known that the categories of left H-modules and left H_F-modules are isomorphic. We have a similar property for Doi-Hopf modules.
Let $(\mathfrak{A}, \rho, \Phi_\rho)$ be a right H-comodule algebra, and let $\Phi_{\rho_F} = (1_{\mathfrak{A}} \otimes F)\Phi_\rho$. Then $\mathfrak{A}_F := (\mathfrak{A}, \rho, \Phi_{\rho_F})$ is a right H_F-comodule algebra, see [14].
Now let C be a left H-module coalgebra, and define a new comultiplication $\underline{\Delta}_F$ as follows: $\underline{\Delta}_F(c) = F\underline{\Delta}(c)$, for all $c \in C$. Then straightforward computations show that $(C, \underline{\Delta}_F, \underline{\varepsilon})$ is a left H_F-module coalgebra and that the categories ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ and ${}_{\mathfrak{A}_F}\mathcal{M}(H_F)^{C_F}$ are isomorphic. Of course, a similar result holds for left comodule algebras.

3.2. **Yetter-Drinfeld modules and Doi-Hopf modules.** Our next aim is to show that the category of left-right Yetter-Drinfeld modules ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is isomorphic to the category of Doi-Hopf modules ${}_{\mathbb{A}^2}\mathcal{M}(H^{\rm op} \otimes H)^C$ and, a fortiori, to ${}_{\mathbb{A}^1}\mathcal{M}(H^{\rm op} \otimes H)^C$, by Proposition 3.4. We have divided the proof over a few lemmas.

Lemma 3.6. *Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. We have a functor*

$$F:\ {}_{\mathbb{A}}\mathcal{YD}(H)^C \to {}_{\mathbb{A}^2}\mathcal{M}(H^{\rm op} \otimes H)^C$$

which acts as the identity on objects an morphisms. If M is a left-right $(H, \mathbb{A}, C)$-Yetter-Drinfeld module then $F(M) = M$ as a left $\mathbb{A}$-module, and with the newly defined right C-coaction

$$\rho'_M(m) = m_{(0)'} \otimes m_{(1)'} = (\tilde{p}^2_\lambda \cdot m)_{(0)} \otimes (\tilde{p}^2_\lambda \cdot m)_{(1)} \cdot \tilde{p}^1_\lambda, \tag{3.15}$$

for all $m \in M$. Here $\tilde{p}_\lambda = \tilde{p}^1_\lambda \otimes \tilde{p}^2_\lambda$ is the element defined by (1.25).

Proof. It is not hard to see that (1.33) and (1.5) imply

$$\Theta^2_{[0]}\tilde{p}^2_\lambda \otimes \Theta^2_{[-1]}\tilde{p}^1_\lambda S^{-1}(\Theta^1) \otimes \Theta^3 = \theta^2(\tilde{p}^2_\lambda)_{\langle 0\rangle} \otimes \theta^1\tilde{p}^1_\lambda \otimes \theta^3(\tilde{p}^2_\lambda)_{\langle 1\rangle}. \tag{3.16}$$

We now show that $F(M)$ satisfies the relations (2.17-2.19). Let $\tilde{P}^1_\lambda \otimes \tilde{P}^2_\lambda$ be another copy of $\tilde{p}_\lambda$, and compute

$$\begin{aligned}
&\Phi_{\rho_2} \cdot (\rho'_M \otimes id_C)(\rho_M(m))\\
&\overset{(3.14,3.15,3.6)}{=} (\tilde{x}^3_\lambda)_{\langle 0\rangle}\tilde{X}^1_\rho\Theta^2_{\langle 0\rangle} \cdot (\tilde{P}^2_\lambda \cdot (\tilde{p}^2_\lambda \cdot m)_{(0)})_{(0)}\\
&\qquad \otimes (\tilde{x}^3_\lambda)_{\langle 1\rangle_1}\tilde{X}^2_\rho\Theta^2_{\langle 1\rangle} \cdot (\tilde{P}^2_\lambda \cdot (\tilde{p}^2_\lambda \cdot m)_{(0)})_{(1)} \cdot \tilde{P}^1_\lambda S^{-1}(f^2\tilde{x}^2_\lambda\Theta^1)\\
&\qquad \otimes (\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\Theta^3 \cdot (\tilde{p}^2_\lambda \cdot m)_{(1)} \cdot \tilde{p}^1_\lambda S^{-1}(f^1\tilde{x}^1_\lambda)\\
&\overset{(3.8,3.16)}{=} (\tilde{x}^3_\lambda)_{\langle 0\rangle}\tilde{X}^1_\rho \cdot (\theta^2(\tilde{P}^2_\lambda)_{\langle 0\rangle} \cdot (\tilde{p}^2_\lambda \cdot m)_{(0)})_{(0)}\\
&\qquad \otimes (\tilde{x}^3_\lambda)_{\langle 1\rangle_1}\tilde{X}^2_\rho \cdot (\theta^2(\tilde{P}^2_\lambda)_{\langle 0\rangle} \cdot (\tilde{p}^2_\lambda \cdot m)_{(0)})_{(1)} \cdot \theta^1\tilde{P}^1_\lambda S^{-1}(f^2\tilde{x}^2_\lambda)\\
&\qquad \otimes (\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\tilde{X}^3_\rho\theta^3(\tilde{P}^2_\lambda)_{\langle 1\rangle} \cdot (\tilde{p}^2_\lambda \cdot m)_{(1)} \cdot \tilde{p}^1_\lambda S^{-1}(f^1\tilde{x}^1_\lambda)
\end{aligned}$$

$$\begin{aligned}
&\overset{(3.8,3.7)}{=} && (\tilde{x}^3_\lambda)_{\langle 0\rangle}\cdot(\tilde{y}^3_\lambda(\tilde{P}^2_\lambda)_{[0]}\tilde{p}^2_\lambda\cdot m)_{(0)}\\
&&&\otimes(\tilde{x}^3_\lambda)_{\langle 1\rangle_1}\cdot(\tilde{y}^3_\lambda(\tilde{P}^2_\lambda)_{[0]}\tilde{p}^2_\lambda\cdot m)_{(1)_{\underline{1}}}\cdot\tilde{y}^1_\lambda\tilde{P}^1_\lambda S^{-1}(f^2\tilde{x}^2_\lambda)\\
&&&\otimes(\tilde{x}^3_\lambda)_{\langle 1\rangle_2}\cdot(\tilde{y}^3_\lambda(\tilde{P}^2_\lambda)_{[0]}\tilde{p}^2_\lambda\cdot m)_{(1)_{\underline{2}}}\cdot\tilde{y}^2_\lambda(\tilde{P}^2_\lambda)_{[-1]}\tilde{p}^1_\lambda S^{-1}(f^1\tilde{x}^1_\lambda)\\
&\overset{(3.3,3.8,1.30)}{=} && (\tilde{p}^2_\lambda\cdot m)_{(0)}\otimes(\tilde{p}^2_\lambda\cdot m)_{(1)_{\underline{1}}}\cdot(\tilde{p}^1_\lambda)_1\otimes(\tilde{p}^2_\lambda\cdot m)_{(1)_{\underline{2}}}\cdot(\tilde{p}^1_\lambda)_2\\
&\overset{(3.3,3.15)}{=} && m_{(0)'}\otimes m_{(1)'_{\underline{1}}}\otimes m_{(1)'_{\underline{2}}}=(id_M\otimes\underline{\Delta})(\rho'_M(m)),
\end{aligned}$$

for all $m\in M$, as needed. The relation (2.18) is trivial and (2.19) follows from (1.26), (3.13) and (3.6). □

Lemma 3.7. *Let H be a quasi-Hopf algebra, C an H-bimodule coalgebra and $\mathbb{A}$ an H-bicomodule algebra. Then we have a functor*

$$G:\ {}_{\mathbb{A}^2}\mathcal{M}(H^{\rm op}\otimes H)^C\to{}_{\mathbb{A}}\mathcal{YD}(H)^C$$

which acts as the identity on objects and morphisms. Let M be a left-right $(H^{\rm op}\otimes H,\mathbb{A}^2,C)$-Hopf module, with left $\mathbb{A}$-action $\cdot$ and right C-coaction ρ'_M, $\rho'_M(m)=m_{(0)'}\otimes m_{(1)'}\in M\otimes C$. Then $G(M)=M$ as a left $\mathbb{A}$-module, with new right C-coaction $\overline{\rho}_M:\ M\to M\otimes C$, given by the formula

$$\overline{\rho}_M(m)=m_{\overline{(0)}}\otimes m_{\overline{(1)}}=(\tilde{q}^2_\lambda)_{\langle 0\rangle}\cdot m_{(0)'}\otimes(\tilde{q}^2_\lambda)_{\langle 1\rangle}\cdot m_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda),\tag{3.17}$$

for all $m\in M$. Here $\tilde{q}_\lambda=\tilde{q}^1_\lambda\otimes\tilde{q}^2_\lambda$ is the element defined in (1.25).

Proof. The most difficult part is tho show that $G(M)$ satisfies the relations (3.7) and (3.8). M is a left-right $(H,\mathbb{A}^2,C)$-Hopf module, so we have by (2.17), (3.13, 3.14) and (1.14):

$$\begin{aligned}
&\tilde{X}^1_\rho((\tilde{x}^3_\lambda)_{\langle 0\rangle}\Theta^2)_{\langle 0\rangle}\cdot m_{(0,0)'}\otimes\tilde{X}^2_\rho((\tilde{x}^3_\lambda)_{\langle 0\rangle}\Theta^2)_{\langle 1\rangle}\cdot m_{(0,1)'}\cdot S^{-1}(f^2\tilde{x}^2_\lambda\Theta^1)\\
&\otimes\tilde{X}^3_\rho(\tilde{x}^3_\lambda)_{\langle 1\rangle}\Theta^3\cdot m_{(1)'}\cdot S^{-1}(f^1\tilde{x}^1_\lambda)=m_{(0)'}\otimes m_{(1)'_{\underline{1}}}\otimes m_{(1)'_{\underline{2}}},
\end{aligned}\tag{3.18}$$

$$(u\cdot m)_{(0)'}\otimes(u\cdot m)_{(1)'}=u_{[0]_{\langle 0\rangle}}\cdot m_{(0)'}\otimes u_{[0]_{\langle 1\rangle}}\cdot m_{(1)'}\cdot S^{-1}(u_{[-1]}),\tag{3.19}$$

for all $m\in M$ and $u\in\mathbb{A}$. Also, (1.33) and (1.5) imply that

$$S(\theta^1)\tilde{q}^1_\lambda\theta^2_{[-1]}\otimes\tilde{q}^2_\lambda\theta^2_{[0]}\otimes\theta^3=\tilde{q}^1_\lambda\Theta^1\otimes(\tilde{q}^2_\lambda)_{\langle 0\rangle}\Theta^2\otimes(\tilde{q}^2_\lambda)_{\langle 1\rangle}\Theta^3.\tag{3.20}$$

Let $\tilde{Q}^1_\lambda\otimes\tilde{Q}^2_\lambda$ be another copy of $\tilde{q}_\lambda$; for all $m\in M$, we compute that

$$\begin{aligned}
&(\theta^2\cdot m_{\overline{(0)}})_{\overline{(0)}}\otimes(\theta^2\cdot m_{\overline{(0)}})_{\overline{(1)}}\otimes\theta^3\cdot m_{\overline{(1)}}\\
&\overset{(3.17,3.19)}{=}(\tilde{Q}^2_\lambda\theta^2_{[0]}(\tilde{q}^2_\lambda)_{\langle 0\rangle_{[0]}})_{\langle 0\rangle}\cdot m_{(0,0)'}\otimes(\tilde{Q}^2_\lambda\theta^2_{[0]}(\tilde{q}^2_\lambda)_{\langle 0\rangle_{[0]}})_{\langle 1\rangle}\cdot m_{(0,1)'}\\
&\qquad\cdot S^{-1}(\tilde{Q}^1_\lambda\theta^2_{[-1]}(\tilde{q}^2_\lambda)_{\langle 0\rangle_{[-1]}})\theta^1\otimes\theta^3(\tilde{q}^2_\lambda)_{\langle 1\rangle}\cdot m_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda)\\
&\overset{(3.20,1.32)}{=}((\tilde{Q}^2_\lambda(\tilde{q}^2_\lambda)_{[0]})_{\langle 0\rangle}\Theta^2)_{\langle 0\rangle}\cdot m_{(0,0)'}\otimes((\tilde{Q}^2_\lambda(\tilde{q}^2_\lambda)_{[0]})_{\langle 0\rangle}\Theta^2)_{\langle 1\rangle}\cdot m_{(0,1)'}\\
&\qquad\cdot S^{-1}(\tilde{Q}^1_\lambda(\tilde{q}^2_\lambda)_{[-1]}\Theta^1)\otimes(\tilde{Q}^2_\lambda(\tilde{q}^2_\lambda)_{[0]})_{\langle 1\rangle}\Theta^3\cdot m_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(1.31,3.18,1.10)}{=} && (\tilde{q}^2_\lambda(\tilde{x}^3_\lambda)_{[0]})_{\langle 0,0\rangle}\tilde{x}^1_\rho\cdot m_{(0)'}\\
&&&\otimes(\tilde{q}^2_\lambda(\tilde{x}^3_\lambda)_{[0]})_{\langle 0,1\rangle}\tilde{x}^2_\rho\cdot m_{(1)'_{\underline{1}}}\cdot S^{-1}(\tilde{q}^1_\lambda(\tilde{x}^3_\lambda)_{[-1]})_1\tilde{x}^1_\lambda\\
&&&\otimes(\tilde{q}^2_\lambda(\tilde{x}^3_\lambda)_{[0]})_{\langle 1\rangle}\tilde{x}^3_\rho\cdot m_{(1)'_{\underline{2}}}\cdot S^{-1}(\tilde{q}^1_\lambda(\tilde{x}^3_\lambda)_{[-1]})_2\tilde{x}^2_\lambda\\
&\overset{(1.14,1.33,3.19)}{=} && \tilde{x}^1_\rho\cdot((\tilde{q}^2_\lambda)_{\langle 0\rangle}\cdot(\tilde{x}^3_\lambda\cdot m)_{(0)'})\\
&&&\otimes\tilde{x}^2_\rho\cdot\left((\tilde{q}^2_\lambda)_{\langle 1\rangle}\cdot(\tilde{x}^3_\lambda\cdot m)_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda)\right)_{\underline{1}}\cdot\tilde{x}^1_\lambda\\
&&&\otimes\tilde{x}^3_\rho\cdot\left((\tilde{q}^2_\lambda)_{\langle 1\rangle}\cdot(\tilde{x}^3_\lambda\cdot m)_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda)\right)_{\underline{2}}\cdot\tilde{x}^2_\lambda\\
&\overset{(3.17)}{=} && \tilde{x}^1_\rho\cdot(\tilde{x}^3_\lambda\cdot m)_{\overline{(0)}}\otimes\tilde{x}^2_\rho\cdot(\tilde{x}^3_\lambda\cdot m)_{\overline{(1)}_{\underline{1}}}\cdot\tilde{x}^1_\lambda\otimes\tilde{x}^3_\rho\cdot(\tilde{x}^3_\lambda\cdot m)_{\overline{(1)}_{\underline{2}}}\cdot\tilde{x}^2_\lambda,
\end{aligned}
$$

as needed. (3.8) also holds since

$$
\begin{aligned}
&u_{\langle 0\rangle}\cdot m_{\overline{(0)}}\otimes u_{\langle 1\rangle}\cdot m_{\overline{(1)}}\\
&\overset{(3.17,1.27)}{=} (\tilde{q}^2_\lambda u_{[0,0]})_{\langle 0\rangle}\cdot m_{(0)'}\otimes(\tilde{q}^2_\lambda u_{[0,0]})_{\langle 1\rangle}\cdot m_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda u_{[0,-1]})u_{[-1]}\\
&\overset{(3.19)}{=} (\tilde{q}^2_\lambda)_{\langle 0\rangle}\cdot(u_{[0]}\cdot m)_{(0)'}\otimes(\tilde{q}^2_\lambda)_{\langle 1\rangle}\cdot(u_{[0]}\cdot m)_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda)u_{[-1]}\\
&\overset{(3.17)}{=} (u_{[0]}\cdot m)_{\overline{(0)}}\otimes(u_{[0]}\cdot m)_{\overline{(1)}}\cdot u_{[-1]},
\end{aligned}
$$

for all $u\in\mathbb{A}$ and $m\in M$. The remaining details are left to the reader. □

Theorem 3.8 is the main result of this Section, and generalizes [8, Theorem 2.3] to the quasi-Hopf algebra setting.

Theorem 3.8. *Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. Then the categories ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ and ${}_{\mathbb{A}^2}\mathcal{M}(H^{\rm op}\otimes H)^C$ are isomorphic. In particular ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is a Grothendieck category, and therefore it has enough injective objects.*

Proof. We show that the functors F and G from Lemmas 3.6 and 3.7 are inverses. Since F and G act as the identity functor at the level of $\mathbb{A}$-modules we have only to show that F and G are inverses at the level of C-coactions.
Let $M\in{}_{\mathbb{A}}\mathcal{YD}(H)^C$ and $\rho_M(m)=m_{(0)}\otimes m_{(1)}$ its right C-coaction. We denote by $\overline{\rho}_M(m)=m_{\overline{(0)}}\otimes m_{\overline{(1)}}$ the right C-coaction of $G(F(M))$ obtained using first Lemma 3.6 and then Lemma 3.7. For all $m\in M$ we then have

$$
\begin{aligned}
\overline{\rho}_M(m) &\overset{(3.17)}{=} (\tilde{q}^2_\lambda)_{\langle 0\rangle}\cdot m_{(0)'}\otimes(\tilde{q}^2_\lambda)_{\langle 1\rangle}\cdot m_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda)\\
&\overset{(3.15)}{=} (\tilde{q}^2_\lambda)_{\langle 0\rangle}\cdot(\tilde{p}^2_\lambda\cdot m)_{(0)}\otimes(\tilde{q}^2_\lambda)_{\langle 1\rangle}\cdot(\tilde{p}^2_\lambda\cdot m)_{(1)}\cdot\tilde{p}^1_\lambda S^{-1}(\tilde{q}^1_\lambda)\\
&\overset{(3.8)}{=} ((\tilde{q}^2_\lambda)_{[0]}\tilde{p}^2_\lambda\cdot m)_{(0)}\otimes((\tilde{q}^2_\lambda)_{[0]}\tilde{p}^2_\lambda\cdot m)_{(1)}\cdot(\tilde{q}^2_\lambda)_{[-1]}\tilde{p}^1_\lambda S^{-1}(\tilde{q}^1_\lambda)\\
&\overset{(1.28)}{=} m_{(0)}\otimes m_{(1)}=\rho_M(m).
\end{aligned}
$$

Conversely, let $M\in{}_{\mathbb{A}^2}\mathcal{M}(H^{\rm op}\otimes H)^C$ and denote by $\rho'_M(m)=m_{(0)'}\otimes m_{(1)'}$ its right C-coaction. If $\rho_M(m)=m_{(0)}\otimes m_{(1)}$ is the right C-coaction on $F(G(M))$ obtained using first Lemma 3.7 and then Lemma 3.6 we have

$$
\begin{aligned}
\rho_M(m) &\overset{(3.15)}{=} (\tilde{p}^2_\lambda\cdot m)_{\overline{(0)}}\otimes(\tilde{p}^2_\lambda\cdot m)_{\overline{(1)}}\cdot\tilde{p}^1_\lambda\\
&\overset{(3.17)}{=} (\tilde{q}^2_\lambda)_{\langle 0\rangle}\cdot(\tilde{p}^2_\lambda\cdot m)_{(0)'}\otimes(\tilde{q}^2_\lambda)_{\langle 1\rangle}\cdot(\tilde{p}^2_\lambda\cdot m)_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda)\tilde{p}^1_\lambda\\
&\overset{(3.19)}{=} (\tilde{q}^2_\lambda(\tilde{p}^2_\lambda)_{[0]})_{\langle 0\rangle}\cdot m_{(0)'}\otimes(\tilde{q}^2_\lambda(\tilde{p}^2_\lambda)_{[0]})_{\langle 1\rangle}\cdot m_{(1)'}\cdot S^{-1}(\tilde{q}^1_\lambda(\tilde{p}^2_\lambda)_{[-1]})\tilde{p}^1_\lambda\\
&\overset{(1.29)}{=} m_{(0)'}\otimes m_{(1)'}=\rho'_M(m),
\end{aligned}
$$

for all $m \in M$, so the proof is finished. $\square$

Let H be a quasi-bialgebra, $\mathfrak{A}$ a right H-comodule algebra and C a left H-module coalgebra. Identifying the category of left-right $(H, \mathfrak{A}, C)$-Hopf modules to the category of right-left $(H^{\mathrm{op,cop}}, \underline{\mathfrak{A}}^{\mathrm{op}}, C^{\mathrm{cop}})$-Hopf modules using the construction preceding Proposition 2.2, we obtain the functor, after permuting the tensor factors:

$$\mathcal{F}' = \bullet \otimes C : \ {}_{\mathfrak{A}}\mathcal{M} \to {}_{\mathfrak{A}}\mathcal{M}(H)^C.$$

If M is a left $\mathfrak{A}$-module then $\mathcal{F}'(M) = M \otimes C$ with structure maps

$$\begin{aligned} \mathfrak{a} \cdot (m \otimes c) &= \mathfrak{a}_{\langle 0 \rangle} \cdot m \otimes \mathfrak{a}_{\langle 1 \rangle} \cdot c, \\ \rho_M(m \otimes c) &= \tilde{x}^1_\rho \cdot m \otimes \tilde{x}^2_\rho \cdot c_{\underline{1}} \otimes \tilde{x}^3_\rho \cdot c_{\underline{2}}, \end{aligned}$$

for all $\mathfrak{a} \in \mathfrak{A}$, $m \in M$ and $c \in C$. For a morphism υ in ${}_{\mathfrak{A}}\mathcal{M}$, we have that $\mathcal{F}'(\upsilon) = \upsilon \otimes id_C$. In particular, we obtain that $\mathfrak{A} \otimes C$ is a left-right $(H, \mathfrak{A}, C)$-Hopf module.
Moreover, $\mathcal{F}'$ is a right adjoint of the forgetful functor $\mathcal{U}^C : \ {}_{\mathfrak{A}}\mathcal{M}(H)^C \to {}_{\mathfrak{A}}\mathcal{M}$, and it is a left adjoint of the functor $\mathrm{Hom}^C_{\mathfrak{A}}(\mathfrak{A} \otimes C, \bullet) : \ {}_{\mathfrak{A}}\mathcal{M}(H)^C \to {}_{\mathfrak{A}}\mathcal{M}$ defined as follows. For $M \in {}_{\mathfrak{A}}\mathcal{M}(H)^C$, $\mathrm{Hom}^C_{\mathfrak{A}}(\mathfrak{A} \otimes C, \bullet)(M) = \mathrm{Hom}^C_{\mathfrak{A}}(\mathfrak{A} \otimes C, M)$, the set of morphisms in ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ between $\mathfrak{A} \otimes C$ and M, viewed as a left $\mathfrak{A}$-module via

$$(\mathfrak{a} \cdot \eta)(\mathfrak{a}' \otimes c) = \eta(\mathfrak{a}'\mathfrak{a} \otimes c),$$

for all $\eta \in \mathrm{Hom}^C_{\mathfrak{A}}(\mathfrak{A} \otimes C, M)$, $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{A}$ and $c \in C$. The functor $\mathrm{Hom}^C_{\mathfrak{A}}(\mathfrak{A} \otimes C, \bullet)$ sends a morphism κ from ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ to the morphism $\vartheta \mapsto \kappa \circ \vartheta$.

Corollary 3.9. *Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. We have a functor $\mathfrak{F} = \bullet \otimes C : \ {}_{\mathbb{A}}\mathcal{M} \to {}_{\mathbb{A}}\mathcal{YD}(H)^C$. The structure maps on $\mathfrak{F}(M) = M \otimes C$ are the following:*

$$\begin{aligned} u \cdot (m \otimes c) &= u_{[0]_{\langle 0 \rangle}} \cdot m \otimes u_{[0]_{\langle 1 \rangle}} \cdot c \cdot S^{-1}(u_{[-1]}), \\ \rho_{M \otimes C}(m \otimes c) &= \theta^2_{\langle 0 \rangle} \tilde{x}^1_\rho ((\tilde{q}^2_\lambda)_{[0]} \tilde{X}^3_\lambda)_{\langle 0 \rangle} \cdot m \otimes \theta^2_{\langle 1 \rangle} \tilde{x}^2_\rho ((\tilde{q}^2_\lambda)_{[0]} \tilde{X}^3_\lambda)_{\langle 1 \rangle_1} \cdot c_{\underline{1}} \\ &\quad \cdot S^{-1}(\theta^1 (\tilde{q}^2_\lambda)_{[-1]} \tilde{X}^2_\lambda g^2) \otimes \theta^3 \tilde{x}^3_\rho ((\tilde{q}^2_\lambda)_{[0]} \tilde{X}^3_\lambda)_{\langle 1 \rangle_2} \cdot c_{\underline{2}} \cdot S^{-1}(\tilde{q}^1_\lambda \tilde{X}^1_\lambda g^1), \end{aligned}$$

for all $u \in \mathbb{A}$, $m \in M$ and $c \in C$. In particular, $\mathbb{A} \otimes C$ is a left-right $(H, \mathbb{A}, C)$-Yetter-Drinfeld module. Moreover, the following assertions hold:

i) *$\mathfrak{F}$ is right adjoint to the forgetful functor $\mathfrak{U}^C : \ {}_{\mathbb{A}}\mathcal{YD}(H)^C \to {}_{\mathbb{A}}\mathcal{M}$.*
ii) *$\mathfrak{F}$ is left adjoint to the functor $\mathrm{Hom}^C_{\mathbb{A}}(\mathbb{A} \otimes C, \bullet) : \ {}_{\mathbb{A}}\mathcal{YD}(H)^C \to {}_{\mathbb{A}}\mathcal{M}$. If $M \in {}_{\mathbb{A}}\mathcal{YD}(H)^C$ then $\mathrm{Hom}^C_{\mathbb{A}}(\mathbb{A} \otimes C, \bullet)(M) = \mathrm{Hom}^C_{\mathbb{A}}(\mathbb{A} \otimes C, M)$, the set of Yetter-Drinfeld morphisms from $\mathbb{A} \otimes C$ to M, viewed as a right $\mathbb{A}$-module via the action*

$$(u \cdot \eta)(u' \otimes c) = \eta(u'u \otimes c),$$

for all $\eta \in \mathrm{Hom}^C_{\mathbb{A}}(\mathbb{A} \otimes C, M)$, $u, u' \in \mathbb{A}$ and $c \in C$.

Proof. The functor $\mathfrak{F}$ is well defined because it is the composition

$$\mathfrak{F} : {}_{\mathbb{A}}\mathcal{M} = {}_{\mathbb{A}^2}\mathcal{M} \xrightarrow{\mathcal{F}'} {}_{\mathbb{A}^2}\mathcal{M}(H^{\mathrm{op}} \otimes H)^C \xrightarrow{G} {}_{\mathbb{A}}\mathcal{YD}(H)^C,$$

where $\mathcal{F}'$ is the functor described above but now for the context given by (3.13-3.14), and G is the functor from Lemma 3.7. All the others details are left to the reader. $\square$

Let H be a quasi-bialgebra, $\mathfrak{A}$ a right H-comodule algebra and C a left H-module coalgebra. Since the category ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ can be identified to a category of right-left Doi-Hopf modules it follows that ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ is isomorphic to a category of comodules over the coring $\mathcal{C}' = C \otimes \mathfrak{A}$, with $\mathfrak{A}$-bimodule structure given by

$$\mathfrak{a} \cdot (c \otimes \mathfrak{a}') \cdot \mathfrak{a}'' = \mathfrak{a}_{\langle 1\rangle} \cdot c \otimes \mathfrak{a}_{\langle 0\rangle}\mathfrak{a}'\mathfrak{a}''$$

and comultiplication and counit given by

$$\Delta_{\mathcal{C}'}(c \otimes \mathfrak{a}) = (\tilde{x}^3_\rho \cdot c_{\underline{2}} \otimes 1_{\mathfrak{A}}) \otimes_{\mathfrak{A}} (\tilde{x}^2_\rho \cdot c_{\underline{1}} \otimes \tilde{x}^1_\rho \mathfrak{a}),\ \varepsilon_{\mathcal{C}'}(c \otimes \mathfrak{a}) = \underline{\varepsilon}(c)\mathfrak{a}$$

for all $\mathfrak{a}, \mathfrak{a}', \mathfrak{a}'' \in \mathfrak{A}$ and $c \in C$. Using arguments similar to the ones given in [3, Theorem 5.4], one can easily check that ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ is isomorphic to ${}^{\mathcal{C}'}\mathcal{M}$.

Corollary 3.10. *Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. Then the category of left-right Yetter-Drinfeld modules ${}_{\mathbb{A}}\mathcal{Y}D(H)^C$ is isomorphic to the category of left comodules over the coring $\mathfrak{C}' = C \otimes \mathbb{A}$, with the following structure maps. The $\mathbb{A}$-bimodule is given by*

$$u \cdot (c \otimes u') \cdot u'' = u_{[0]_{\langle 1\rangle}} \cdot c \cdot S^{-1}(u_{[-1]}) \otimes u_{[0]_{\langle 0\rangle}} u'u'',$$

and the comultiplication and counit are defined by the formulas

$$\begin{aligned}\Delta_{\mathfrak{C}'}(c \otimes u) = &\left(\theta^3 \tilde{x}^3_\rho (\tilde{X}^3_\lambda)_{\langle 1\rangle_2} \cdot c_{\underline{2}} \cdot S^{-1}(\tilde{X}^1_\lambda g^1) \otimes 1_{\mathbb{A}}\right)\\ &\otimes_{\mathbb{A}} \left(\theta^2_{\langle 1\rangle} \tilde{x}^2_\rho (\tilde{X}^3_\lambda)_{\langle 1\rangle_1} \cdot c_{\underline{1}} \cdot S^{-1}(\theta^1 \tilde{X}^2_\lambda g^2) \otimes \theta^2_{\langle 0\rangle} \tilde{x}^1_\rho (\tilde{X}^3_\lambda)_{\langle 0\rangle} u\right),\\ \varepsilon_{\mathfrak{C}'}(c \otimes u) = &\ \underline{\varepsilon}(c)u,\end{aligned}$$

for all $u, u', u'' \in \mathbb{A}$ and $c \in C$.

Proof. This is a direct consequence of the above comments and Theorem 3.8. □

3.3. **The category of Yetter-Drinfeld modules as a module category.** Our next aim is to describe the category of Yetter-Drinfeld modules as a category of modules. We will need the (right) generalized diagonal crossed product construction, as introduced in [14, 7].

Let H be a quasi-bialgebra and $\mathbb{A}$ an H-bicomodule algebra. In the sequel, $\mathcal{A}$ will be an H-bimodule algebra. This means that $\mathcal{A}$ is an H-bimodule which has a multiplication and a usual unit $1_{\mathcal{A}}$ such that for all $\varphi, \psi, \xi \in \mathcal{A}$ and $h \in H$ the following relations hold:

$$\begin{aligned}&(\varphi\psi)\xi = (X^1 \cdot \varphi \cdot x^1)[(X^2 \cdot \psi \cdot x^2)(X^3 \cdot \xi \cdot x^3)],\\ &h \cdot (\varphi\psi) = (h_1 \cdot \varphi)(h_2 \cdot \psi),\ (\varphi\psi) \cdot h = (\varphi \cdot h_1)(\psi \cdot h_2),\\ &h \cdot 1_{\mathcal{A}} = \varepsilon(h)1_{\mathcal{A}},\ 1_{\mathcal{A}} \cdot h = \varepsilon(h)1_{\mathcal{A}}.\end{aligned}$$

If H is a quasi-bialgebra, then H^*, the linear dual of H, is an H-bimodule via the H-actions

$$\langle h \rightharpoonup \varphi, h'\rangle = \varphi(h'h),\ \langle \varphi \leftharpoonup h, h'\rangle = \varphi(hh'),$$

for all $\varphi \in H^*$, $h, h' \in H$. The convolution $\langle \varphi\psi, h\rangle = \varphi(h_1)\psi(h_2)$, $\varphi, \psi \in H^*$, $h \in H$, is a multiplication on H^* which is not associative in general, but with this multiplication H^* becomes an H-bimodule algebra.

Let H be a quasi-bialgebra, $\mathcal{A}$ an H-bimodule algebra and $(\mathbb{A}, \lambda, \rho, \Phi_\lambda, \Phi_\rho, \Phi_{\lambda,\rho})$ an H-bicomodule algebra. In the sequel (δ, Ψ) will be the pair

$$\begin{aligned}&\delta_l = (\lambda \otimes id_H) \circ \rho,\\ &\Psi_: = (id_H \otimes \lambda \otimes id_H^{\otimes 2})\left((\Phi_{\lambda,\rho} \otimes 1_H)(\lambda \otimes id_H^{\otimes 2})(\Phi_\rho^{-1})\right)[\Phi_\lambda \otimes 1_H^{\otimes 2}],\end{aligned}$$

or

$$\delta_r = (id_H \otimes \rho) \circ \lambda,$$
$$\Psi_r = (id_H^{\otimes 2} \otimes \rho \otimes id_H)\left((1_H \otimes \Phi_{\lambda,\rho}^{-1})(id_H^{\otimes 2} \otimes \rho)(\Phi_\lambda)\right)[1_H^{\otimes 2} \otimes \Phi_\rho^{-1}].$$

$\Omega_{L_{l/r}}, \Omega_{R_{l/r}} \in H^{\otimes 2} \otimes \mathbb{A} \otimes H^{\otimes 2}$ are defined by the following formulas

$$\Omega_{L_{l/r}} = (id_H^{\otimes 2} \otimes id_{\mathbb{A}} \otimes S^{-1} \otimes S^{-1})(\Psi_{l/r}^{-1}) \cdot [1_H^{\otimes 2} \otimes 1_{\mathbb{A}} \otimes S^{-1}(f^1) \otimes S^{-1}(f^2)],$$
$$\Omega_{R_{l/r}} = [S^{-1}(g^1) \otimes S^{-1}(g^2) \otimes 1_{\mathbb{A}} \otimes 1_H^{\otimes 2}] \cdot (S^{-1} \otimes S^{-1} \otimes id_{\mathbb{A}} \otimes id_H^{\otimes 2})(\Psi_{l/r}).$$

Here $f = f^1 \otimes f^2$ is the Drinfeld twist and $f^{-1} = g^1 \otimes g^2$ is its inverse. We will use the notation

$$\Omega_{L_{l/r}} = \Omega^1_{L_{l/r}} \otimes \cdots \otimes \Omega^5_{L_{l/r}}.$$

Let $\mathcal{A} \bowtie_{\delta_l} \mathbb{A}$ and $\mathcal{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} \mathbb{A}$ be the vector space $\mathcal{A} \otimes \mathbb{A}$ furnished with the multiplication given respectively by the following formulas:

$$(3.21) \quad (\varphi \bowtie u)(\psi \bowtie u') = (\Omega^1_{L_l} \cdot \varphi \cdot \Omega^5_{L_l})(\Omega^2_{L_l} u_{\langle 0\rangle_{[-1]}} \cdot \psi \cdot S^{-1}(u_{\langle 1\rangle})\Omega^4_{L_l}) \bowtie \Omega^3_{L_l} u_{\langle 0\rangle_{[0]}} u',$$

$$(3.22) \quad (\varphi \blacktriangleright\!\!\blacktriangleleft u)(\psi \blacktriangleright\!\!\blacktriangleleft u') = (\Omega^1_{L_r} \cdot \varphi \cdot \Omega^5_{L_r})(\Omega^2_{L_r} u_{[-1]} \cdot \psi \cdot S^{-1}(u_{[0]_{\langle 1\rangle}})\Omega^4_{L_r}) \blacktriangleright\!\!\blacktriangleleft \Omega^3_{L_r} u_{[0]_{\langle 0\rangle}} u',$$

for all $\varphi, \psi \in \mathcal{A}$ and $u, u' \in \mathbb{A}$. We write $\varphi \bowtie u$ (respectively $\varphi \blacktriangleright\!\!\blacktriangleleft u$) for $\varphi \otimes u$ considered as an element of $\mathcal{A} \bowtie_{\delta_l} \mathbb{A}$ (respectively $\mathcal{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} \mathbb{A}$). $\mathcal{A} \bowtie_{\delta_l} \mathbb{A}$ and $\mathcal{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} \mathbb{A}$ are isomorphic associative algebras with unit $1_{\mathcal{A}} \bowtie 1_{\mathbb{A}}$ (respectively $1_{\mathcal{A}} \blacktriangleright\!\!\blacktriangleleft 1_{\mathbb{A}}$), containing $\mathbb{A} \cong 1_{\mathcal{A}} \bowtie \mathbb{A}$ (respectively $\mathbb{A} \cong 1_{\mathcal{A}} \blacktriangleright\!\!\blacktriangleleft \mathbb{A}$) as unital subalgebra. These algebras are called the left generalized diagonal crossed products.
The right generalized diagonal crossed products are introduced in a similar way: denote

$$\Omega_{R_{l/r}} = \Omega^1_{R_{l/r}} \otimes \cdots \otimes \Omega^5_{R_{l/r}},$$

and let $\mathbb{A} \bowtie_{\delta_l} \mathcal{A}$ and $\mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} \mathcal{A}$ be the vector space $\mathbb{A} \otimes \mathcal{A}$ with the following product:

$$(3.23) \quad (u \bowtie \varphi)(u' \bowtie \psi) = uu'_{\langle 0\rangle_{[0]}} \Omega^3_{R_l} \bowtie (\Omega^2_{R_l} S^{-1}(u'_{\langle 0\rangle_{[-1]}}) \cdot \varphi \cdot u'_{\langle 1\rangle}\Omega^4_{R_l})(\Omega^1_{R_l} \cdot \psi \cdot \Omega^5_{R_l}),$$

$$(3.24) \quad (u \blacktriangleright\!\!\blacktriangleleft \varphi)(u' \blacktriangleright\!\!\blacktriangleleft \psi) = uu'_{[0]_{\langle 0\rangle}} \Omega^3_{R_r} \blacktriangleright\!\!\blacktriangleleft (\Omega^2_{R_r} S^{-1}(u'_{[-1]}) \cdot \varphi \cdot u'_{[0]_{\langle 1\rangle}}\Omega^4_{R_r})(\Omega^1_{R_r} \cdot \psi \cdot \Omega^5_{R_r}),$$

for all $u, u' \in \mathbb{A}$ and $\varphi, \psi \in \mathcal{A}$. $\mathbb{A} \bowtie_{\delta_l} \mathcal{A}$ and $\mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} \mathcal{A}$ are isomorphic associative algebras with unit $1_{\mathbb{A}} \bowtie 1_{\mathcal{A}}$ and $1_{\mathbb{A}} \blacktriangleright\!\!\blacktriangleleft 1_{\mathcal{A}}$, containing $\mathbb{A}$ as a unital subalgebra.
As algebras, the left and right generalized crossed product algebras are isomorphic, see [14, 7]. If H is a quasi-Hopf algebra then $\mathcal{A} = H^*$ is an H-bimodule algebra. In this particular case the left and right generalized diagonal crossed products are exactly the left and the right diagonal crossed products constructed in [14]. In this way Hausser and Nill gave four explicit realizations of $D(H)$, the quantum double of a finite dimensional quasi-Hopf algebra H. Two of them are build on $H^* \otimes H$ and the other two on $H \otimes H^*$. All these are, as algebras, diagonal crossed products. The first two realizations built on $H^* \otimes H$ coincide to $H^* \bowtie_{\delta_l} H$ and $H^* \blacktriangleright\!\!\blacktriangleleft_{\delta_r} H$, and the last two realizations built on $H \otimes H^*$ coincide to $H \bowtie_{\delta_l} H^*$ and $H \blacktriangleright\!\!\blacktriangleleft_{\delta_r} H^*$.

Proposition 3.11. *Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. Then $\mathbb{A}^1 \mathbin{>\!\!\blacktriangleleft} C^* \equiv \mathbb{A} \bowtie_{\delta_l} C^*$ and $\mathbb{A}^2 \mathbin{>\!\!\blacktriangleleft} C^* \equiv \mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} C^*$, as algebras. In particular, the algebras $\mathbb{A}^1 \mathbin{>\!\!\blacktriangleleft} C^*$ and $\mathbb{A}^2 \mathbin{>\!\!\blacktriangleleft} C^*$ are isomorphic to each other, and also to the four generalized diagonal crossed products.*

Proof. Keeping the above concepts and notations it is not hard to see that for an H-bicomodule algebra $\mathbb{A}$ the reassociators Φ_{ρ_1} and Φ_{ρ_2} defined in (3.12) and (3.14), respectively, can be rewritten as

$$\Phi_{\rho_1} = \tilde{\Omega}^3_{R_l} \otimes (\tilde{\Omega}^2_{R_l} \otimes \tilde{\Omega}^4_{R_l}) \otimes (\tilde{\Omega}^1_{R_l} \otimes \tilde{\Omega}^5_{R_l}),$$
$$\Phi_{\rho_2} = \tilde{\Omega}^3_{R_r} \otimes (\tilde{\Omega}^2_{R_r} \otimes \tilde{\Omega}^4_{R_r}) \otimes (\tilde{\Omega}^1_{R_r} \otimes \tilde{\Omega}^5_{R_r}),$$

where we used the notation

$$\Omega^{-1}_{R_{l/r}} = \tilde{\Omega}^1_{R_{l/r}} \otimes \cdots \otimes \tilde{\Omega}^5_{R_{l/r}}.$$

Now, if C is an H-bimodule coalgebra viewed as a left $H^{\rm op} \otimes H$-module coalgebra via the structure defined in (3.6) then C^*, the linear dual space of C, is a right $H^{\rm op} \otimes H$-module algebra. The multiplication of C^* is the convolution, that is $(c^*d^*)(c) = c^*(c_{\underline{1}})d^*(c_{\underline{2}})$, the unit is $\underline{\varepsilon}$ and the right $H^{\rm op} \otimes H$-module action is given by the formula $(c^* \cdot (h \otimes h'))(c) = c^*(h' \cdot c \cdot h) = (h \rightharpoonup c^* \leftharpoonup h')(c)$, for all $h, h' \in H$, $c^*, d^* \in C^*$, $c \in C$.
Since $\mathbb{A}^{1/2}$ are right $H^{\rm op} \otimes H$-comodule algebras and C^* is a right $H^{\rm op} \otimes H$-module coalgebra, it makes sense to consider the right generalized smash product algebras $\mathbb{A}^{1/2} \mathbin{>\!\!\blacktriangleleft} C^*$ (cf. Remark 2.13 i)). It also follows easily from Remark 2.13 i) that the multiplication on $\mathbb{A}^1 \mathbin{>\!\!\blacktriangleleft} C^*$ is given by

$$\begin{aligned}
&(u \mathbin{>\!\!\blacktriangleleft} c^*)(u' \mathbin{>\!\!\blacktriangleleft} d^*)\\
&\quad = uu'_{\langle 0\rangle_{[0]}}\Omega^3_{R_l} \otimes \Big((c^* \cdot (\Omega^2_{R_l}S^{-1}(u'_{\langle 0\rangle_{[-1]}}) \otimes u'_{\langle 1\rangle}\Omega^4_{R_l}))(d^* \cdot (\Omega^1_{R_l} \otimes \Omega^5_{R_l}))\Big)\\
&\quad = uu'_{\langle 0\rangle_{[0]}}\Omega^3_{R_l} \otimes (\Omega^2_{R_l}S^{-1}(u'_{\langle 0\rangle_{[-1]}}) \rightharpoonup c^* \leftharpoonup u'_{\langle 1\rangle}\Omega^4_{R_l})(\Omega^1_{R_l} \rightharpoonup d^* \leftharpoonup \Omega^5_{R_l}).
\end{aligned}$$

On the other hand, it is easy to see that the linear dual space C^* of an H-bimodule coalgebra C is an H-bimodule algebra. The multiplication of C^* is the convolution, the unit is $\underline{\varepsilon}$ and the H-bimodule structure is given by the formula $(h \rightharpoonup c^* \leftharpoonup h')(c) = c^*(h' \cdot c \cdot h)$, for all $h, h' \in H$, $c^* \in C^*$, $c \in C$. So we can consider the generalized right diagonal crossed product $\mathbb{A} \bowtie_{\delta_l} C^*$. From (3.23) it follows that the multiplication rule on $\mathbb{A} \bowtie_{\delta_l} C^*$ coincides with the multiplication of $\mathbb{A}^1 \mathbin{>\!\!\blacktriangleleft} C^*$, hence the algebras $\mathbb{A} \bowtie_{\delta_l} C^*$ and $\mathbb{A}^1 \mathbin{>\!\!\blacktriangleleft} C^*$ are equal. In a similar way we can show the equality of the k-algebras $\mathbb{A}^2 \mathbin{>\!\!\blacktriangleleft} C^*$ and $\mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} C^*$. □

Remark 3.12. It was shown in [7] that the left generalized crossed product algebras $\mathcal{A} \bowtie_{\delta_l} \mathbb{A}$ and $\mathcal{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} \mathbb{A}$ coincide with the left generalized smash product algebras $\mathcal{A} \blacktriangleright\!\!< \mathbb{A}_1$ and $\mathcal{A} \blacktriangleright\!\!< \mathbb{A}_2$, respectively. The generalized smash products are made over $H \otimes H^{\rm op}$ and by $\mathbb{A}_1$ and $\mathbb{A}_2$ we denote the left $H \otimes H^{\rm op}$-comodule algebras structures on $\mathbb{A}$ defined at the end of Section 1.

Let H be a quasi-bialgebra, $\mathfrak{A}$ a right H-comodule algebra and C a left H-module algebra. Viewing the category ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ as a category of right-left Doi-Hopf modules, we deduce from Theorem 2.6 that ${}_{\mathfrak{A}}\mathcal{M}(H)^C$ is isomorphic to the category of rational $\mathfrak{A} \mathbin{>\!\!\blacktriangleleft} C^*$-modules, $\mathrm{Rat}({}_{\mathfrak{A} \mathbin{>\!\!\blacktriangleleft} C^*}\mathcal{M})$, and that $\mathrm{Rat}({}_{\mathfrak{A} \mathbin{>\!\!\blacktriangleleft} C^*}\mathcal{M}) = \sigma_{\mathfrak{A} \mathbin{>\!\!\blacktriangleleft} C^*}[\mathfrak{A} \otimes C]$.

A rational $\mathfrak{A} \succ\!\!\blacktriangleleft C^*$-module is a left $\mathfrak{A} \succ\!\!\blacktriangleleft C^*$-module M such that for any $m \in M$ there exist two finite families $\{c_i\}_i \subseteq C$ and $\{m_i\}_i \subseteq M$ such that

$$(\mathfrak{a} \succ\!\!\blacktriangleleft c^*) \cdot m = c^*(c_i)(\mathfrak{a} \succ\!\!\blacktriangleleft \underline{\varepsilon}) \cdot m_i,$$

for all $\mathfrak{a} \in \mathfrak{A}$ and $c^* \in C^*$.

Corollary 3.13. *Let H be a quasi-Hopf algebra, $\mathbb{A}$ an H-bicomodule algebra and C an H-bimodule coalgebra. Then the following assertions hold:*

- i) *The category of left-right Yetter-Drinfeld modules ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is isomorphic to the category of rational $\mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} C^*$-modules $\mathrm{Rat}({}_{\mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} C^*}\mathcal{M})$, which is also equal to the category $\sigma_{\mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} C^*}[\mathbb{A} \otimes C]$.*
- ii) *If C is finite dimensional then ${}_{\mathbb{A}}\mathcal{YD}(H)^C$ is isomorphic to the category of left $C^* \bowtie_{\delta_l} \mathbb{A}$-modules.*

Proof. The assertion i) follows easily from the above comments and Theorem 3.8. ii) If C is finite dimensional then $\mathrm{Rat}({}_{\mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} C^*}\mathcal{M}) = {}_{\mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} C^*}\mathcal{M}$. Moreover, $\mathbb{A} \blacktriangleright\!\!\blacktriangleleft_{\delta_r} C^*$ is always isomorphic to $C^* \bowtie_{\delta_l} \mathbb{A}$ as an algebra. We note that another proof of this result can be found in [7]. □

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